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Attribute Reduction in Bipolar Intuitionistic Fuzzy Relation System

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Abstract: Bipolar intuitionistic fuzzy relation system (BIFRS) effectively characterize the bipolarity in real-world data by introducing positive and negative membership and non-membership degrees. To address redundancy caused by high-dimensional attributes, this paper proposes an attribute reduction method based on bipolar intuitionistic discernibility matrices. By constructing positive discernibility matrix M^{+*} and negative discernibility matrix M^{-*}, we systematically identify the minimal attribute subset $O \subseteq \mathcal{Q}$ that satisfies the bipolar consistency condition $h^{+} = O = h^{-}$, where h^{+} and h^{-} are discernibility functions derived from their disjunctive normal forms. Theoretical proofs demonstrate that the global relation $S_{\mathcal{Q}}$ and reduced relation S_O maintain complete consistency across all four components $(\pi^N, \pi^P, \eta^N, \eta^P)$ if and only if O covers all positive and negative discernibility terms. Furthermore, we develop an efficient reduction algorithm that reduces time complexity through optimized matrix operations. Experimental results on real-world datasets validate the algorithm's effectiveness and superiority. This work not only extends the reduction theories of traditional fuzzy sets and rough sets, but also provides interpretable and efficient tools for decision support and feature selection applications.

Keywords: Bipolar Intuitionistic Fuzzy Sets, Attribute Reduction, Discernibility Matrix

Introduction

In today's era of big data and intricate systems, managing uncertainty and bipolarity in information poses a fundamental challenge across diverse domains, including artificial intelligence, decision-making, network analysis, and beyond. The traditional fuzzy set theory, introduced by Zadeh (1965), laid the groundwork for addressing fuzziness through membership degrees. The swift advancement of fuzzy set theory has solidified its position as a central research focus in mathematics, engineering, computer science, signal processing, graph theory, and numerous other disciplines. However, Zadeh's classical fuzzy set theory has limitations in accommodating both supportive and opposing information concurrently. Atanassov's introduction of intuitionistic fuzzy sets (IFS) (Atanassov, 1999) expanded the theory by incorporating membership, non-membership, and hesitation degrees, thereby enhancing the flexibility of uncertainty representation. Xu (2012) developed intuitionistic fuzzy interactive decision making for dynamic multi-criteria systems. Lei et al. (2014) significantly extended the theoretical framework by combining Borel measurable spaces with intuitionistic fuzzy sets, resolving the compatibility issues between probabilistic and fuzzy measures. Yet, these approaches still grapple with bipolarity-pthe coexistence of opposing concepts like positive/negative or benefit/risk-within a unified framework. Zhang (1998) introduced the concept of bipolar fuzzy sets as a generalization of fuzzy sets, and Massa'deh and collaborators (Massa'deh, 2010, 2013, 2015, 2017; Ismail & Massa'deh, 2013; Massa'deh & Fora, 2017) introduced many new concepts, including fuzzy and bipolar fuzzy.

Bipolar Fuzzy Sets (BFS) have extended traditional fuzzy sets to bipolar scales of $[-1, 1]$, revolutionizing the modeling of contradictory attributes. In a bipolar fuzzy set, positive information signifies what is guaranteed to be possible, while negative information denotes what is impossible or forbidden, or rarely false. Lee (2000) introduced another generalization of fuzzy sets where the membership degree is expanded from the interval $[0, 1]$

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Despite significant progress, the practical application of Bipolar Intuitionistic Fuzzy Relation System (BIFRS) still faces one major challenge: high-dimensional redundancy. BIFRS often contains overlapping or irrelevant attributes, which leads to compromised computational efficiency and diminished model interpretability. To tackle this fundamental issue, this paper introduces the framework of BIFRS and develops a novel attribute reduction algorithm. Our method employs bipolar intuitionistic fuzzy relation sets to construct discernibility matrices and determines optimal reduction sets through minimal disjunctive normal forms. Finally, the effectiveness and practicality of the proposed algorithm are verified through case analysis.

Preliminaries

Bipolar intuitionistic fuzzy sets ($BIFS$) form the foundation of our study on bipolar intuitionistic fuzzy relation systems. The symbol $\wedge$ denotes conjunction, $\vee$ denotes disjunction, $\cup$ denotes union, and $\cap$ denotes intersection. The key definitions and properties are outlined below.

Definition 1 (Ezhilmaran and Sankar, 2015)

A Bipolar Intuitionistic Fuzzy Set ($BIFS$) on a universe is defined as:

$$S = \{\langle u, \pi_S^N(u), \pi_S^P(u), \eta_S^N(u), \eta_S^P(u)\rangle | u \in \mathcal{U}\},$$

Where $\pi_S^N(u), \eta_S^N(u) : \mathcal{U} \to [-1, 0], \pi_S^P(u), \eta_S^P(u) : \mathcal{U} \to [0, 1]$ such that $0 \le \pi_S^P + \eta_S^P \le 1,\ -1 \le \pi_S^N + \eta_S^N \le 0$.

Similar to how binary relations on a set X are represented by subsets of the Cartesian product $X \times X$, bipolar intuitionistic fuzzy relations are constructed using a comparable methodology.

Definition 2 (Ezhilmaran and Sankar, 2015)

A bipolar intuitionistic fuzzy relation $S = (\pi_S^N, \pi_S^P, \eta_S^N, \eta_S^P)$ over a universe $\mathcal{U}$ is characterized as a bipolar intuitionistic fuzzy set ($BIFS$) on the Cartesian product $\mathcal{U} \times \mathcal{U}$, which can be formally represented as follows:

$$S = \{\langle (u, v), \pi_S^N(u, v), \pi_S^P(u, v), \eta_S^N(u, v), \eta_S^P(u, v)\rangle | (u, v) \in \mathcal{U} \times \mathcal{U}\}$$

Therefore, π_S^N and η_S^N are mappings from $\mathcal{U} \times \mathcal{U}$ to the interval $[-1, 0]$, while π_S^P and η_S^P are mappings from $[0, 1]$ to the interval.

The notion of bipolar intuitionistic fuzzy sets can be elaborated as follows. The positive membership degree $\pi^P(u)$ quantifies the extent to which an element u satisfies the property associated with the bipolar intuitionistic fuzzy set S. Conversely, the negative membership degree $\pi^N(u)$ reflects the degree to which u fulfills the implicit counter-property linked to S. Meanwhile, the positive non-membership degree $\eta^P(u)$ indicates the level of non-membership with respect to S,

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and the negative non-membership degree $\eta^N(u)$ captures the extent to which u does not satisfy the implicit counter-property corresponding to the set.

Thus, the relationship from u to v is expressed as $S(u,v) = (\pi_S^N(u,v), \pi_S^P(u,v), \eta_S^N(u,v), \eta_S^P(u,v))$. Compared to the elementary information provided by a binary relation R, where the alternative notation $(x,y) \in R$ or $R(x,y)$ merely indicates that " x is related to y ", this representation offers a more convenient framework for knowledge expression. For example, consider a simple product preference model involving a user u and a product v. Let the evaluation of user u towards product v be represented by a bipolar intuitionistic fuzzy value $(\pi^N, \pi^P, \eta^N, \eta^P)$. Here, (π^N, π^P) and (η^N, η^P) represent the degrees to which user u likes and dislikes product v, respectively, while (π^N, η^N) and (π^P, η^P) signify the level of certainty associated with these respective degrees of liking and disliking. A value of $(-0.8, 0.95, -0.1, 0.02)$ clearly indicates that user u likes product v.

Definition 3 (Ezhilmaran and Sankar, 2015)

Let S be a bipolar intuitionistic fuzzy relation (BIF relation) defined on the universe of discourse $\mathcal{U}$. Then,

1. S is called reflexive if $\pi_S^P(u,u) = 1$, $\pi_S^N(u,u) = 0$, and $\eta_S^P(u,u) = 1$, $\eta_S^N(u,u) = 0$ for each $u \in \mathcal{U}$.
2. S is called symmetric if $\pi_S^N(u,v) = \pi_S^N(v,u)$, $\pi_S^P(u,v) = \pi_S^P(v,u)$, and $\eta_S^N(u,v) = \eta_S^N(v,u)$, $\eta_S^P(u,v) = \eta_S^P(v,u)$ for each $u, v \in \mathcal{U}$.
3. S is called transitive if $\pi_S^N(u,v) = \pi_S^N(u,w)$ and $\pi_S^N(u,w) = \pi_S^N(v,w)$, then $\pi_S^N(u,v) = \pi_S^N(v,w)$; and if $\pi_S^P(u,v) = \pi_S^P(u,w)$ and $\pi_S^P(u,w) = \pi_S^P(v,w)$, then $\pi_S^P(u,v) = \pi_S^P(v,w)$ for all $u, v, w \in \mathcal{U}$. Moreover, if $\eta_S^N(u,v) = \eta_S^N(u,w)$ and $\eta_S^N(u,w) = \eta_S^N(v,w)$, then $\eta_S^N(u,v) = \eta_S^N(v,w)$; and if $\eta_S^P(u,v) = \eta_S^P(u,w)$ and $\eta_S^P(u,w) = \eta_S^P(v,w)$ then $\eta_S^P(u,v) = \eta_S^P(v,w)$ for all $u, v, w \in \mathcal{U}$.

Definition 4 (Ezhilmaran and Sankar, 2015)

Let $S = \{\pi_S^N(u), \pi_S^P(u), \eta_S^N(u), \eta_S^P(u)\}$ and $S' = \{\pi_{S'}^N(u), \pi_{S'}^P(u), \eta_{S'}^N(u), \eta_{S'}^P(u)\}$ be bipolar intuitionistic fuzzy sets on a set X. If $S = \{\pi_S^N(u), \pi_S^P(u),$
$\eta_S^N(u), \eta_S^P(u)\}$ is a bipolar intuitionistic fuzzy relation on $S' = \{\pi_{S'}^N(u), \pi_{S'}^P(u), \eta_{S'}^N(u), \eta_{S'}^P(u)\}$ if
$\pi_S^N(u,v) \geq \pi_{S'}^N(u) \vee \pi_{S'}^N(v), \pi_S^P(u,v) \leq \pi_{S'}^P(u) \wedge \pi_{S'}^P(v),$
$\eta_S^N(u,v) \leq \eta_{S'}^N(u) \wedge \eta_{S'}^N(v), \eta_S^P(u,v) \geq \eta_{S'}^P(u) \vee \eta_{S'}^P(v),$
for all $x, y \in X$.

The combination of union and intersection for bipolar intuitionistic fuzzy relations (BIF relations) is defined as follows:

Definition 5 (Ezhilmaran and Sankar, 2015)

Consider a finite collection $\{S_k\}_{k \in I}$ of bipolar intuitionistic fuzzy relations (BIF relations) over a universe $\mathcal{U}$. Each relation S_k, where $k \in I$, is represented as $S_k = (\pi_{S_k}^N, \pi_{S_k}^P, \eta_{S_k}^N, \eta_{S_k}^P)$. The intersection $\cap_{k \in I} S_k$ and the union $\cup_{k \in I} S_k$ are also BIF relations on $\mathcal{U}$, defined according to the following expressions:

1. $\cap_{k \in I} S_k(u,v) = (\vee_{k \in I} \pi_{S_k}^N(u,v), \wedge_{k \in I} \pi_{S_k}^P(u,v), \wedge_{k \in I} \eta_{S_k}^N(u,v), \vee_{k \in I} \eta_{S_k}^P(u,v))$ for each $u, v \in \mathcal{U}$.
2. $\cup_{k \in I} S_k(u,v) = (\wedge_{k \in I} \pi_{S_k}^N(u,v), \vee_{k \in I} \pi_{S_k}^P(u,v), \vee_{k \in I} \eta_{S_k}^N(u,v), \wedge_{k \in I} \eta_{S_k}^P(u,v))$ for each $u, v \in \mathcal{U}$.

Additionally, when S and S' are BIF relations defined on $\mathcal{U}$, if for all $u, v \in \mathcal{U}$, $\pi_S^P(u,v) \leq \pi_{S'}^P(u,v)$, $\pi_S^N(u,v) \geq \pi_{S'}^N(u,v)$, $\eta_S^P(u,v) \geq \eta_{S'}^P(u,v)$, and $\eta_S^N(u,v) \leq \eta_{S'}^N(u,v)$, then we denote $S \subseteq S'$. The symbols $\wedge$ and $\vee$ represent the minimum and maximum, respectively.

It should be noted that for a binary relation S defined on the set $\mathcal{U} = \{u_1, u_2, \cdots, u_n\}$, the corresponding relation matrix M_S is an $n \times n$ matrix where each element a_{ij} is determined by $S(u_i, u_j)$. This results in $a_{ij} = S(u_i, u_j) = (\pi_S^N(u_i, u_j), \pi_S^P(u_i, u_j), \eta_S^N(u_i, u_j), \eta_S^P(u_i, u_j))$.

Next, we review some fundamental concepts, namely information systems, discernibility matrices, and discernibility functions.

Definition 6 (Skowron & Rauszer, 1992)

An information system can be defined as a couple $(\mathcal{U}, \mathcal{Q})$, where $\mathcal{U}$ denotes a nonempty universe set and $\mathcal{Q}$ represents a collection of attributes. Each attribute q in $\mathcal{Q}$ maps elements from $\mathcal{U}$ to its corresponding value set V_q, signifying the range of values that the attribute can take.

Definition 7 (Skowron & Rauszer, 1992)

Let $\mathcal{U} = \{u_1, u_2, \ldots, u_n\}$ denote the universal set and $\mathcal{Q} = \{q_1, q_2, \ldots, q_m\}$ represent the attribute set. In the context of an information system $(\mathcal{U}, \mathcal{Q})$, the discernibility matrix $M(\mathcal{Q}) = (c_{ij})$ is an $n \times n$ matrix defined as follows:

$c_{ij} = \{q \in \mathcal{Q} | q(u_i) \vDash q(q(u_j)), i, j = 1, 2, \cdots, n\}.$

Definition 8 (Skowron & Rauszer, 1992)

In an information system $(\mathcal{U}, \mathcal{Q})$, the discernibility function $h_{\mathcal{Q}}$ is a Boolean expression involving m

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Boolean variables $\hat{q}_1, \ldots, \hat{q}_m$, each corresponding to an attribute from the set $q_1, \ldots, q_m$. It is defined as follows:

$h_{\mathcal{Q}}(\hat{q}_1, \cdots, \hat{q}_m) = \wedge\{\vee(c_{ij}) | 1 \leq j \leq i \leq n, c_{ij} \vDash \emptyset\}$.

Now we define $BIFRS$.

Definition 9

Let $\mathcal{U}$ be a nonempty finite universe of discourse, and let $\mathcal{Q} = \{S_1, S_2, \ldots, S_m\}$ represent a family of binary relations defined on $\mathcal{U}$, referred to as bipolar intuitionistic fuzzy relations.

The following remarks elucidate the importance of Definition 8 by relating it to existing concepts in the literature:

Remark 1 (Ali et al., 2020)

The notions introduced in Definition 8 generalize several well-known concepts as special cases.

1. If $\mathcal{Q}$ is composed of equivalence relations on $\mathcal{U}$, then $(\mathcal{U}, \mathcal{Q})$ constitutes an information system.
2. If $\mathcal{Q}$ comprises binary relations on $\mathcal{U}$, then $(\mathcal{U}, \mathcal{Q})$ forms a relation system.
3. If $\mathcal{Q}$ consists of fuzzy relations on $\mathcal{U}$, then $(\mathcal{U}, \mathcal{Q})$ corresponds to a fuzzy relation system.

Definition 10

Let $(\mathcal{U}, \mathcal{Q})$ be a binary relation system ($BIFS$) according to the notation in Definition 8. For any non-empty subset $\emptyset \vDash O \subseteq \mathcal{Q}$, we define a binary relation (BIF relation) $\mathcal{U}$ on S_O as follows for all $u_i, u_j \in \mathcal{U}$, according to Definition 5.

$$\begin{aligned} S_O(u_i, u_j) &= \bigcap_{S_k \in O} S_k \\ &= \left(\bigvee_{S_k \in O} \pi^N_{S_k}(u_i, u_j), \bigwedge_{S_k \in O} \pi^P_{S_k}(u_i, u_j), \right. \\ &\left. \bigwedge_{S_k \in O} \eta^N_{S_k}(u_i, u_j), \bigvee_{S_k \in O} \eta^P_{S_k}(u_i, u_j) \right), \\ &\forall u_i, u_j \in \mathcal{U}, \end{aligned}$$

Definition 10 builds a binary relation (BIF relation) related to $(\mathcal{U}, \mathcal{Q})$ by extracting the information from a subset O of the original binary relations (BIF relations) and merging it into a unified binary relation (BIF relation).

Attribute Reduction on BIFRS

The objective of this paper is to introduce the following new concept:

Definition 11

Consider a $BIFRS(\mathcal{U}, \mathcal{Q})$ and a subset $O \subseteq \mathcal{Q}$. If the following conditions hold:

1. For all $u_i, u_j \in \mathcal{U}$, $S_{\mathcal{Q}}(u_i, u_j) = S_O(u_i, u_j)$.
2. For each non-empty subset $O' \subseteq O$, $S_{\mathcal{Q}} \vDash S_{O'}$.

In this context, the set O is termed an attribute reduct of $\mathcal{Q}$.

Analogous to the representation of knowledge via discernibility matrices in information systems, we extend this formulation to bipolar intuitionistic fuzzy relation systems ($BIFRS$). Consider a $BIFRS$ denoted by $(\mathcal{U}, \mathcal{Q})$, where $\mathcal{U} = \{u_1, u_2, \cdots, u_n\}$ is a finite universe of discourse.

To compute the reduction, we introduce the positive and negative discernibility matrices $M^+ = (s^+_{ij})_{n\times n}$ and $M^- = (s^-_{ij})_{n\times n}$, defined for $i, j \in I$ with $I = \{1, 2, \ldots, n\}$ as follows:

$$\begin{aligned} s^+_{ij} = \{S \in \mathcal{Q} \mid \pi^P_S(u_i, u_j) = \pi^P_{S_{\mathcal{Q}}}(u_i, u_j), \\ \eta^P_S(u_i, u_j) = \eta^P_{S_{\mathcal{Q}}}(u_i, u_j)\}, \end{aligned} \tag{a}$$

$$\begin{aligned} s^-_{ij} = \{S \in \mathcal{Q} \mid \pi^N_S(u_i, u_j) = \pi^N_{S_{\mathcal{Q}}}(u_i, u_j), \\ \eta^N_S(u_i, u_j) = \eta^N_{S_{\mathcal{Q}}}(u_i, u_j)\}. \end{aligned} \tag{b}$$

Thus, each cell of this matrix aggregates the binary relations (BIF relations) that capture the exact information obtained by merging the attributes relevant to the corresponding pair of elements. The positive and negative discernibility functions are respectively defined as $h^- = \bigwedge_{s^-_{ij} \vDash \emptyset} \left(\bigvee s^-_{ij}\right)$ and $h^+ = \bigwedge_{s^+_{ij} \vDash \emptyset} \left(\bigvee s^+_{ij}\right)$.

To establish an attribute reduction algorithm, this paper initiates the discussion through the following perspectives:

Lemma 1

Let $(\mathcal{U}, \mathcal{Q})$ be a $BIFRS$. Then for all $i, j \in I$, s^+_{ij} and s^-_{ij} are non-empty sets.

Proof

Since $\mathcal{Q}$ is finite, for $u_i, u_j \in \mathcal{U}$, there exists at least one $S_k \in \mathcal{Q}$ such that $S_k(u_i, u_j) = S_{\mathcal{Q}}(u_i, u_j)$. Therefore, $S_k \in s^+_{ij}$ and $s^+_{ij} \vDash \emptyset$. Similarly, it can be proven that $s^-_{ij} \vDash \emptyset$.

Theorem 1

Consider a $BIFRS(\mathcal{U}, \mathcal{Q})$ with a given non-empty subset $O \subseteq \mathcal{Q}$. The following properties hold:

1. $\pi^P_{S_{\mathcal{Q}}}(u_i, u_j) = \pi^P_{S_O}(u_i, u_j), \eta^P_{S_{\mathcal{Q}}}(u_i, u_j) = \eta^P_{S_O}(u_i, u_j)$
   if and only if $s^+_{ij} \cap O \vDash \emptyset, \forall i, j \in I$.
2. $\pi^N_{S_{\mathcal{Q}}}(u_i, u_j) = \pi^N_{S_O}(u_i, u_j), \eta^N_{S_{\mathcal{Q}}}(u_i, u_j) = \eta^N_{S_O}(u_i, u_j)$

if and only if $s_{ij}^{-} \cap O \vDash \emptyset, \forall i, j \in I$.

Proof

1. Suppose for all $u_i, u_j \in \mathcal{U}$, $\pi_{S_{\mathcal{Q}}}^{P}(u_i, u_j) = \pi_{S_O}^{P}(u_i, u_j), \eta_{S_{\mathcal{Q}}}^{P}(u_i, u_j) = \eta_{S_O}^{P}(u_i, u_j)$. According to Definition 8, $\mathcal{Q}$ is a finite set, so there exists at least one $S_k \in O$ such that $\pi_{S_k}^{P}(u_i, u_j) = \pi_{S_{\mathcal{Q}}}^{P}(u_i, u_j) = \pi_{S_O}^{P}(u_i, u_j), \eta_{S_k}^{P}(u_i, u_j) = \eta_{S_{\mathcal{Q}}}^{P}(u_i, u_j) = \eta_{S_O}^{P}(u_i, u_j)$, Therefore, $S_k \in O \cap s_{ij}^{+}$, which means $s_{ij}^{+} \cap O \vDash \emptyset$. Conversely, assume for all $i, j \in I$, $s_{ij}^{+} \cap O \vDash \emptyset$. Suppose $S_k \in s_{ij}^{+} \cap O$. By the definition of s_{ij}^{+}, $\pi_{S_{\mathcal{Q}}}^{P}(u_i, u_j) = \pi_{S_k}^{P}(u_i, u_j) = \pi_{S_O}^{P}(u_i, u_j), \eta_{S_{\mathcal{Q}}}^{P}(u_i, u_j) = \eta_{S_k}^{P}(u_i, u_j) = \eta_{S_O}^{P}(u_i, u_j)$. Therefore, $\pi_{S_{\mathcal{Q}}}^{P}(u_i, u_j) = \pi_{S_O}^{P}(u_i, u_j), \eta_{S_{\mathcal{Q}}}^{P}(u_i, u_j) = \eta_{S_O}^{P}(u_i, u_j)$.
2. Suppose for all $u_i, u_j \in \mathcal{U}$, $\pi_{S_{\mathcal{Q}}}^{N}(u_i, u_j) = \pi_{S_O}^{N}(u_i, u_j), \eta_{S_{\mathcal{Q}}}^{N}(u_i, u_j) = \eta_{S_O}^{N}(u_i, u_j)$. According to Definition 8, $\mathcal{Q}$ is a finite set, so there exists at least one $S_k \in O$ such that $\pi_{S_k}^{N}(u_i, u_j) = \pi_{S_{\mathcal{Q}}}^{N}(u_i, u_j) = \pi_{S_O}^{N}(u_i, u_j), \eta_{S_k}^{N}(u_i, u_j) = \eta_{S_{\mathcal{Q}}}^{N}(u_i, u_j) = \eta_{S_O}^{N}(u_i, u_j)$. Therefore, $S_k \in O \cap s_{ij}^{-}$, which means $s_{ij}^{-} \cap O \vDash \emptyset$. Conversely, assume for all $i, j \in I$, $s_{ij}^{+} \cap O \vDash \emptyset$. Suppose $S_k \in s_{ij}^{-} \cap O$. By the definition of s_{ij}^{-}, $\pi_{S_{\mathcal{Q}}}^{N}(u_i, u_j) = \pi_{S_k}^{N}(u_i, u_j) = \pi_{S_O}^{N}(u_i, u_j), \eta_{S_{\mathcal{Q}}}^{N}(u_i, u_j) = \eta_{S_k}^{N}(u_i, u_j) = \eta_{S_O}^{N}(u_i, u_j)$. Therefore, $\pi_{S_{\mathcal{Q}}}^{N}(u_i, u_j) = \pi_{S_O}^{N}(u_i, u_j), \eta_{S_{\mathcal{Q}}}^{N}(u_i, u_j) = \eta_{S_O}^{N}(u_i, u_j)$.

Theorem 2

Let $(\mathcal{U}, \mathcal{Q})$ be a $BIFRS$ with a nonempty subset $\emptyset \vDash O \subseteq \mathcal{Q}$. The equality $S_{\mathcal{Q}}(u_i, u_j) = S_O(u_i, u_j)$ holds for all $i, j \in I$ if and only if $h^{+} = O = h^{-}$.

Proof

Let $(\mathcal{U}, \mathcal{Q})$ be a bipolar intuitionistic Fuzzy relation system ($BIFRS$) with $\emptyset \vDash O \subseteq \mathcal{Q}$. Assume $h^{+} = O = h^{-}$. For any (u_i, u_j), there exists $S_K \in O$ such that $\pi_{S_k}^{N}(u_i, u_j) = \bigvee_{S_l \in \mathcal{Q}} \pi_{S_l}^{N}(u_i, u_j), \eta_{S_k}^{N}(u_i, u_j) = \bigwedge_{S_l \in \mathcal{Q}} \eta_{S_l}^{P}(u_i, u_j), \pi_{S_k}^{P}(u_i, u_j) = \bigwedge_{S_l \in \mathcal{Q}} \pi_{S_l}^{P}(u_i, u_j), \eta_{S_k}^{P}(u_i, u_j) = \bigvee_{S_l \in \mathcal{Q}} \eta_{S_l}^{P}(u_i, u_j)$. Thus, O contains at least one attribute that attains the global minimum for π^{P}, η^{N} and the global maximum for π^{N}, η^{P}. Consequently $\bigwedge_{S_k \in O} \pi_{S_k}^{P} = \bigwedge_{S_l \in \mathcal{Q}} \pi_{S_l}^{P}$, $\bigwedge_{S_k \in O} \eta_{S_k}^{N} = \bigwedge_{S_l \in \mathcal{Q}} \eta_{S_l}^{N}$, $\bigvee_{S_k \in O} \pi_{S_k}^{N} = \bigvee_{S_l \in \mathcal{Q}} \pi_{S_l}^{N}$, $\bigvee_{S_k \in O} \eta_{S_k}^{P} = \bigvee_{S_l \in \mathcal{Q}} \eta_{S_l}^{P}$. This implies $\pi_{S_{\mathcal{Q}}}^{P} = \bigwedge \pi_{S_l}^{P} = \bigwedge \pi_{S_k}^{P} = \pi_{S_O}^{P}$, $\pi_{S_{\mathcal{Q}}}^{N} = \bigvee \pi_{S_l}^{N} = \bigvee \pi_{S_k}^{N} = \pi_{S_O}^{N}$, $\eta_{S_{\mathcal{Q}}}^{N} = \bigwedge \eta_{S_l}^{N} = \bigwedge \eta_{S_k}^{N} = \eta_{S_O}^{N}$, $\eta_{S_{\mathcal{Q}}}^{P} = \bigvee \eta_{S_l}^{P} = \bigvee \eta_{S_k}^{P} = \pi_{S_O}^{P}$. Therefore, $S_{\mathcal{Q}}(u_i, u_j) = S_O(u_i, u_j)$ holds for all u_i, u_j.

Conversely, if $S_{\mathcal{Q}}(u_i, u_j) = S_O(u_i, u_j)$, then for all u_i, u_j, $\bigwedge_O \pi_{S_k}^{P}(u_i, u_j) = \bigwedge_{\mathcal{Q}} \pi_{S_l}^{P}(u_i, u_j), \bigwedge_O \eta_{S_k}^{N}(u_i, u_j) = \bigwedge_{\mathcal{Q}} \eta_{S_l}^{N}(u_i, u_j), \bigvee_O \pi_{S_k}^{N}(u_i, u_j) = \bigvee_{\mathcal{Q}} \pi_{S_l}^{N}(u_i, u_j), \bigvee_O \eta_{S_k}^{P}(u_i, u_j) = \bigvee_{\mathcal{Q}} \eta_{S_l}^{P}(u_i, u_j)$. By Theorem 1, $s_{ij}^{+} \cap O \vDash \emptyset$ and $s_{ij}^{-} \cap O \vDash \emptyset$, meaning O covers all non-empty items in the positive and negative discernibility matrices M^{+} and M^{-}. Hence, the disjunctive normal forms of h^{+} and h^{-} reduce to O, i.e., $h^{+} = O = h^{-}$.

Corollary 1

Let $(\mathcal{U}, \mathcal{Q})$ be a $BIFIS$, and let $O(\vDash \emptyset) \subseteq \mathcal{Q}$. Then, O is a reduct of $\mathcal{Q}$ if and only if O is the smallest subset satisfying $h^{+} = O = h^{-}$.

Proof

If O is a reduction of $\mathcal{Q}$, then $S_{\mathcal{Q}}(u_i, u_j) = S_O(u_i, u_j)$ holds. By Theorem 2, $h^{+} = O = h^{-}$, and O is the minimal subset.

Conversely, if O is the minimal subset satisfying $h^{+} = O = h^{-}$, then by Theorem 2, $h^{+} = O = h^{-}$, and O is both the minimal subset and the reduction.

Given formulas (a) and (b), the computational complexity of determining the discernibility matrices M^{+} and M^{-} stands at $O(mn^2)$. To optimize this process, we introduce new discernibility matrices M^{-*} and M^{+*} as replacements for the original matrices M^{-} and M^{+}. These new matrices offer enhanced computational efficiency. Specifically, we substitute the original matrices $M^{-} = (s_{ij}^{-})_{n \times n}$ and $M^{+} = (s_{ij}^{+})_{n \times n}$ with the newly defined $M^{-*} = (s_{ij}^{-*})_{n \times n}$ and $M^{+*} = (s_{ij}^{+*})_{n \times n}$.

$$s_{ij}^{+*} = \begin{cases} \{S_k | S_k \in \mathcal{Q}, \pi_{S_{\mathcal{Q}}}^{p}(u_i, u_j) = \pi_{S_k}^{p}(u_i, u_j), \eta_{S_{\mathcal{Q}}}^{p}(u_i, u_j) = \eta_{S_k}^{p}(u_i, u_j)\}, & \pi_{S_{\mathcal{Q}}}^{p}(u_i, u_j) \vDash 1, \eta_{S_{\mathcal{Q}}}^{p}(u_i, u_j) \vDash 1, \\ \emptyset, & \text{otherwise.} \end{cases}$$

$$s_{ij}^{-*} = \begin{cases} \{S_k | S_k \in \mathcal{Q}, \pi_{S_{\mathcal{Q}}}^{N}(u_i, u_j) = \pi_{S_k}^{N}(u_i, u_j), \eta_{S_{\mathcal{Q}}}^{N}(u_i, u_j) = \eta_{S_k}^{N}(u_i, u_j)\}, & \pi_{S_{\mathcal{Q}}}^{N}(u_i, u_j) \vDash -1, \eta_{S_{\mathcal{Q}}}^{N}(u_i, u_j) \vDash -1 \\ \emptyset, & \text{otherwise.} \end{cases}$$

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The newly defined discernibility matrices M^{+*} and M^{-*} significantly cut down the computational time compared with the original matrices M^{+} and M^{-}. This is clearly demonstrated in Example 1.

Drawing on Corollary 1, Algorithm 1 has been devised to identify all the reducts within a fuzzy relation system $(BIFRS)$ $(\mathcal{U}, \mathcal{Q})$.

Algorithm 1: Attribute reduction in bipolar intuitionistic fuzzy relation

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Input: Bipolar intuitionistic fuzzy relation system $(\mathcal{U}, \mathcal{Q})$, where $(\mathcal{U} = \{u_1, u_2, \ldots, u_n\}$ is the universe of discourse, and $\mathcal{Q} = \{S_1, S_2, \ldots, S_n\}$ is the set of bipolar intuitionistic fuzzy relations.

Step 1: Calculate the membership and non-membership matrices of the bipolar intuitionistic fuzzy relation according to Definition 10:

$$\pi^N_{S_\mathcal{Q}}(u_i, u_j) = \max_{S_k \in \mathcal{Q}} \pi^N_{S_k}(u_i, u_j)$$

$$\pi^P_{S_\mathcal{Q}}(u_i, u_j) = \min_{S_k \in \mathcal{Q}} \pi^P_{S_k}(u_i, u_j)$$

$$\eta^N_{S_\mathcal{Q}}(u_i, u_j) = \min_{S_k \in \mathcal{Q}} \eta^N_{S_k}(u_i, u_j)$$

$$\eta^P_{S_\mathcal{Q}}(u_i, u_j) = \max_{S_k \in \mathcal{Q}} \eta^P_{S_k}(u_i, u_j)$$

Step 2: Construct the positive and negative discernibility matrices $M^{-*} = (s^{-*}_{ij})_{n \times n}$ and $M^{+*} = (s^{+*}_{ij})_{n \times n}$.

Step 3: Generate the positive and negative discernibility functions h^+ and h^-:

$$h^+ = \bigwedge_{s^{++}_{ij} \vDash \emptyset} \left(\bigvee_{S_k \in s^{+*}_{ij}} S_k \right), h^- = \bigwedge_{s^{--}_{ij} \vDash \emptyset} \left(\bigvee_{S_k \in s^{-*}_{ij}} S_k \right)$$

Step 4: Convert h^+ and h^- from conjunctive normal form to disjunctive normal form:

$$h^+ = \bigvee_{z=1}^{k} \left(\bigwedge O_z \right), \quad O_z \subseteq \mathcal{Q}$$

$$h^- = \bigvee_{x=1}^{l} \left(\bigwedge O_x \right), \quad O_x \subseteq \mathcal{Q}$$

Output: If $h^+ \equiv h^-$, then obtain all minimal reducts $\text{Core}(\mathcal{Q}) = \bigcap_{z=1}^{k} O_z$ according to Definition 12 and $\{O_1, O_2, \ldots, O_k\}$.

The time complexity for computing the extreme values of membership and non-membership degrees for each object (u_i, u_j) is $O(mn^2)$. For each object pair (u_i, u_j), the time complexity for checking whether all attributes satisfy the difference condition is also $O(mn^2)$,but it is closer to $O(n^2)$. The time complexity for performing conjunction and disjunction operations on each non-empty matrix element is $O(mn^2)$. With heuristic optimization strategies, this can be reduced to polynomial time $O(m^3)$. Therefore, the total complexity is $O(mn^2 + m^3)$.

As pointed out in the literature (Leskovec et al., 2011), converting the conjunctive normal form (CNF) of the discernibility function $h = \bigwedge_{s^+_{ij} \vDash \emptyset} (\bigvee s^+_{ij})$ to its disjunctive normal form (DNF) $h = \bigvee_{s^+_{ij} \vDash \emptyset} (\bigwedge s^+_{ij})$ is time-consuming. Borowik and Luba (2014) proposed a fast algorithm to compute this transformation.

Example 1

Consider a universe of discourse $\mathcal{U} = \{u_1, u_2, u_3, u_4\}$ consisting of four objects, and let $\mathcal{Q} = \{S_1, S_2, S_3, S_4\}$ denote a collection of binary relations $(BIF$ relations) defined over $\mathcal{U}$. Each relation is expressed through its corresponding relational matrix, presented as follows:

$$M_1 = \begin{pmatrix} (-0.6, 0.45, -0.4, 0.4) & (-0.4, 0.7, -0.3, 0.5) & (-0.2, 0.5, -0.7, 0.4) & (-0.3, 0.5, -0.4, 0.5) \\ (-0.4, 0.3, -0.3, 0.1) & (-0.2, 0.8, -0.2, 0.5) & (-0.3, 0.5, -0.6, 0.1) & (-0.5, 0.6, -0.4, 0.5) \\ (-0.5, 0.6, -0.3, 0.5) & (-0.5, 0.8, -0.7, 0.4) & (-0.8, 0.5, -0.6, 0.3) & (-0.5, 0.8, -0.4, 0.6) \\ (-0.1, 0.5, -0.8, 0.1) & (-0.3, 0.4, -0.8, 0.1) & (-0.7, 0.4, -0.5, 0.4) & (-0.2, 0.6, -0.8, 0.2) \end{pmatrix},$$

$$M_2 = \begin{pmatrix} (-0.5, 0.5, -0.5, 0.3) & (-0.3, 0.6, -0.4, 0.5) & (-0.4, 0.8, -0.7, 0.1) & (-0.7, 0.9, -0.5, 0.5) \\ (-0.3, 0.7, -0.3, 0.3) & (-0.8, 0.8, -0.2, 0.3) & (-0.3, 0.5, -0.5, 0.1) & (-0.2, 0.4, -0.2, 0.6) \\ (-0.6, 0.8, -0.6, 0.1) & (-0.6, 0.6, -0.5, 0.1) & (-0.8, 0.3, -0.6, 0.2) & (-0.6, 0.9, -0.2, 0.5) \\ (-0.4, 0.7, -0.4, 0.6) & (-0.3, 0.6, -0.7, 0.2) & (-0.7, 0.4, -0.4, 0.2) & (-0.9, 0.7, -0.5, 0.3) \end{pmatrix},$$

$$M_3 = \begin{pmatrix} (-0.6, 0.6, -0.6, 0.3) & (-0.2, 0.4, -0.4, 0.0) & (-0.4, 0.7, -0.6, 0.1) & (-0.5, 0.8, -0.8, 0.2) \\ (-0.3, 0.7, -0.5, 0.1) & (-0.8, 0.3, -0.3, 0.5) & (-0.3, 1.0, -0.2, 0.6) & (-0.2, 0.5, -0.6, 0.2) \\ (-0.6, 0.5, -0.6, 0.1) & (-0.7, 0.6, -0.4, 0.3) & (-0.8, 0.3, -0.7, 0.3) & (-0.7, 0.2, -0.8, 0.2) \\ (-0.4, 0.75, -0.8, 0.1) & (-0.3, 0.4, -0.1, 0.0) & (-0.6, 0.4, -0.3, 0.6) & (-0.9, 0.6, -0.7, 0.1) \end{pmatrix},$$

$$M_4 = \begin{pmatrix} (-0.4, 0.35, -0.5, 0.4) & (-0.2, 0.6, -0.5, 0.0) & (-0.2, 0.7, -0.5, 0.2) & (-0.5, 0.4, -0.5, 0.4) \\ (-0.4, 0.4, -0.7, 0.5) & (-0.2, 0.5, -0.7, 0.2) & (-0.1, 0.8, -0.6, 0.1) & (-0.6, 0.4, -0.6, 0.2) \\ (-0.4, 0.65, -0.7, 0.4) & (-0.7, 0.8, -0.4, 0.4) & (-0.8, 0.7, -0.6, 0.1) & (-0.6, 0.6, -0.4, 0.4) \\ (-0.1, 0.5, -0.3, 0.6) & (-0.3, 0.3, -0.3, 0.6) & (-0.8, 0.6, -0.7, 0.3) & (-0.9, 0.9, -0.3, 0.6) \end{pmatrix}.$$

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Therefore, $(\mathcal{U}, \mathcal{Q})$ is a $BIFRS$, and the relation matrix of $S_{\mathcal{Q}}$ is given by

$$M_{\mathcal{Q}} = \left(\bigvee_{k=1}^{4} \pi_{M_k}^{N}(u_i,u_j), \bigwedge_{k=1}^{4} \pi_{M_k}^{P}(u_i,u_j, \bigwedge_{k=1}^{4} \eta_{M_k}^{N}(u_i,u_j), \bigvee_{k=1}^{4} \eta_{M_k}^{P}(u_i,u_j)\right)$$

So:

$$M_{\mathcal{Q}} = \begin{pmatrix} (-0.4,0.35,-0.6,0.4) & (-0.2,0.4,-0.5,0.5) & (-0.2,0.5,-0.7,0.4) & (-0.3,0.4,-0.8,0.5) \\ (-0.3,0.3,-0.7,0.5) & (-0.2,0.3,-0.7,0.5) & (-0.1,0.5,-0.6,0.6) & (-0.2,0.4,-0.6,0.6) \\ (-0.4,0.5,-0.7,0.5) & (-0.5,0.6,-0.7,0.4) & (-0.8,0.3,-0.7,0.3) & (-0.5,0.2,-0.8,0.6) \\ (-0.1,0.5,-0.8,0.6) & (-0.3,0.3,-0.8,0.6) & (-0.6,0.4,-0.7,0.6) & (-0.2,0.6,-0.8,0.6) \end{pmatrix}.$$

Accordingly, the positive discernibility matrix $M^{+*} = (s_{ij}^{+*})_{4\times 4}$ is presented as follows:

$$M^{+*} = \begin{pmatrix} \{S_1,S_4\} & \{S_1,S_2,S_3\} & \{S_1\} & \{S_1,S_2,S_4\} \\ \{S_1,S_4\} & \{S_1,S_3\} & \{S_1,S_2,S_3\} & \{S_2,S_4\} \\ \{S_1,S_3\} & \mathcal{Q} & \{S_1,S_2,S_3\} & \{S_1,S_3\} \\ \{S_1,S_2,S_4\} & \{S_4\} & \{S_1,S_2,S_3\} & \{S_1,S_3,S_4\} \end{pmatrix}$$

The positive discernibility function h^+ can be expressed as $h^+ = S_1 \wedge S_4$.

Likewise, the negative discernibility matrix $M^{-*} = (s_{ij}^{-*})_{4\times 4}$ can be determined as follows:

$$M^{-*} = \begin{pmatrix} \{S_3,S_4\} & \{S_3,S_4\} & \{S_1,S_2,S_4\} & \{S_1,S_3\} \\ \{S_2,S_3,S_4\} & \{S_1,S_4\} & \{S_1,S_4\} & \{S_2,S_3,S_4\} \\ \{S_4\} & \{S_1\} & \mathcal{Q} & \{S_1,S_3\} \\ \{S_1,S_3,S_4\} & \mathcal{Q} & \{S_3,S_4\} & \{S_1\} \end{pmatrix}$$

The negative discernibility function h^- can be expressed as $h^- = S_1 \wedge S_4$.

It is evident that $h^+ = h^-$. One can readily observe that for $O_1 = \{S_1\} \subseteq \mathcal{Q}$ and $O_2 = \{S_4\} \subseteq \mathcal{Q}$, for all $u_i, u_j \in \mathcal{U}$, we have $S_{O_1}(u_i,u_j) = S_{O_2}(u_i,u_j) = S_{\mathcal{Q}}(u_i,u_j)$. Therefore, $O_1 = \{S_1\}$ and $O_2 = \{S_4\}$ are reducts of $(\mathcal{U}, \mathcal{Q})$. It should be noted that existing attribute reduction algorithms based on discernibility matrices are not capable of handling this example.

Furthermore, a novel reduction algorithm is introduced to extend the applicability of bipolar intuitionistic fuzzy relation systems ($BIFRS$) to set-valued information systems. By employing Algorithm 2, set-valued information systems can be converted into bipolar intuitionistic fuzzy relational matrices through the following formulas:

$$S_k(u_i,u_j) = \left(-\frac{|S_k(u_i)\cap S_k(u_j)|}{|S_k(u_i)\cup S_k(u_j)|}, \frac{|S_k(u_i)\cap S_k(u_j)|}{|S_k(u_i)\cup S_k(u_j)|}, -\frac{|S_k(u_i)\triangle S_k(u_j)|}{|S_k(u_i)\cup S_k(u_j)|}, \frac{|S_k(u_i)\triangle S_k(u_j)|}{|S_k(u_i)\cup S_k(u_j)|}\right).$$

Here, $|S_k(u_i)|$ denotes the cardinality of the set $S_k(u_i)$. $|S_k(u_i)\triangle S_k(u_j)| = |S_k(u_i)\backslash S_k(u_j)| + |S_k(u_j)\backslash S_k(u_i)|$ (Leskovec et al., 2011).

During the initialization phase of the algorithm, it is necessary to store four three-dimensional matrices $(\pi^P, \pi^N, \eta^P, \eta^N)$ of size $n \times n \times m$, resulting in a total space consumption of $4 \times n^2 \times m = O(mn^2)$. The time complexity for parsing input data is $O(mn)$. The computation time complexity for BIFRS relations is $O(mn)$. The time complexity for generating the matrix is $O(n^2)$. The total time complexity is $T(n,m) = O(mn) + O(mn^2) + O(n^2) = O(mn^2)$. The space complexity is $S(n,m) = O(mn^2)$, allowing for real-time computation under conventional scales $(n < 10, m < 10^3)$.

Example 2

Consider the information system $(\mathcal{U}, \mathcal{Q})$ presented in Table 1, with $\mathcal{U} = \{u_1, u_2, \ldots, u_4\}$ and $\mathcal{Q} = \{S_1, S_2, S_3, S_4\}$.

Algorithm 2: Transforming a set-Valued information system into a bipolar intuitionistic fuzzy relation system ($BIFRS$)

Input:
Number of universe elements n
Number of bipolar intuitionistic fuzzy relations m
Binary row vector I (e.g., $\{1,3\} \to 1010$)

Initialization:
Relation matrix $x \leftarrow \text{zeros}(n,m)$
Positive membership $\pi_P \leftarrow \text{zeros}(n,n,m)$
Positive non-membership $\eta_P \leftarrow \text{zeros}(n,n,m)$
Negative membership $\pi_N \leftarrow \text{zeros}(n,n,m)$
Negative non-membership $\eta_N \leftarrow \text{zeros}(n,n,m)$

Parsing Input Data:
For $i = 1$ **to** m **Do**
  For $e = 1$ **to** n **Do**
    $x(e,i) \leftarrow I[(i-1)\cdot n + e]$
  End For
End For

Computing BIF **Relation:**
For $k = 1$ **to** m **Do**
  For $a = 1$**to** n **Do**
    For $j = 1$ **to** n **Do**
      $\text{Intersect} \leftarrow \sum(x(a,k)\ \&\ x(j,k))$
      $\text{Union} \leftarrow \sum(x(a,k) \mid x(j,k))$
      If Union $= 0$ **Then**
        $\pi^P(a,j,k) \leftarrow 0,\ \eta^P(a,j,k) \leftarrow 1$
        $\pi^N(a,j,k) \leftarrow 0,\ \eta^N(a,j,k) \leftarrow 1$
      Else
        $\pi^P(a,j,k) \leftarrow \frac{\text{Intersect}}{\text{Union}}$
        $\eta^P(a,j,k) \leftarrow 1 - \mu_P(a,j,k)$
        $\pi^N(a,j,k) \leftarrow -\mu_P(a,j,k)$
        $\eta^N(a,j,k) \leftarrow 1 - |\mu_N(a,j,k)|$
      End If
    End For
  End For
End For

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Generating BIFRS Matrix:
Step 1: $T \leftarrow \text{zeros}(n, n, 4)$ Comment: Format: $T(i,j) = (\pi^N, \eta^N, \pi^P, \eta^P)$
For $i = 1$ **to** n **Do**
  For $j = 1$ **to** n **Do**
    $T(i, j, :) \leftarrow [\pi^N(i,j), \eta^N(i,j), \pi^P(i,j), \eta^P(i,j)]$
  End For
End For
Output:
  Print the bipolar intuitionistic fuzzy relation matrix T

Table 1: An information table

U	S_1	S_2	S_3	S_4
u_1	$\{1,2\}$	$\{2\}$	$\{1,2,3\}$	$\{2,3\}$
u_2	$\{3\}$	$\{3\}$	$\{1\}$	$\{2\}$
u_3	$\{1,3\}$	$\{2,3\}$	$\{3\}$	$\{1,2\}$
u_4	$\{1,2\}$	$\{3\}$	$\{1,3\}$	$\{1,3\}$

Each S_k can be expressed as a symmetric bipolar intuitionistic fuzzy relation on $\mathcal{U}$. Each symmetric BIF relation $S_k(k = 1,2,3,4)$ is given by their corresponding relation matrices as follows:

$$M_1 = \begin{pmatrix} (-1.00,1.00,0.00,0.00) & (0.00,0.00,-1.00,1.00) & (-0.33,0.33,-0.67,0.67) & (-1.00,1.00,0.00,0.00) \\ (0.00,0.00,-1.00,1.00) & (-1.00,1.00,0.00,0.00) & (-0.50,0.50,-0.50,0.50) & (0.00,0.00,-1.00,1.00) \\ (-0.33,0.33,-0.67,0.67) & (-0.50,0.50,-0.50,0.50) & (-1.00,1.0,00.0,0.00) & (-0.33,0.33,-0.67,0.67) \\ (-1.00,1.00,0.00,0.00) & (0.00,0.00,-1.00,1.00) & (-0.33,0.33,-0.67,0.67) & (-1.00,1.00,0.00,0.00) \end{pmatrix}$$

$$M_2 = \begin{pmatrix} (-1.00,1.00,0.00,0.00) & (0.00,0.00,-1.00,1.00) & (-0.50,0.50,-0.50,0.50) & (0.00,0.00,-1.00,1.00) \\ (0.00,0.00,-1.00,1.00) & (-1.00,1.00,0.00,0.00) & (-0.50,0.50,-0.50,0.50) & (-1.00,1.00,0.00,0.00) \\ (-0.50,0.50,-0.50,0.50) & (-0.50,0.50,-0.50,0.50) & (-1.00,1.00,0.00,0.00) & (-0.50,0.50,-0.50,0.50) \\ (0.00,0.00,-1.00,1.00) & (-1.00,1.00,0.00,0.00) & (-0.50,0.50,-0.50,0.50) & (-1.00,1.00,0.00,0.00) \end{pmatrix}$$

$$M_3 = \begin{pmatrix} (-1.00,1.00,0.00,0.00) & (-0.33,0.33,-0.67,0.67) & (-0.33,0.33,-0.67,0.67) & (-0.67,0.67,-0.33,0.33) \\ (-0.33,0.33,-0.67,0.67) & (-1.00,1.00,0.00,0.00) & (0.00,0.00,-1.00,1.00) & (-0.50,0.50,-0.50,0.50) \\ (-0.33,0.33,-0.67,0.67) & (0.00,0.00,-1.00,1.00) & (-1.00,1.00,0.00,0.00) & (-0.50,0.50,-0.50,0.50) \\ (-0.67,0.67,-0.33,0.33) & (-0.50,0.50,-0.50,0.50) & (-0.50,0.50,-0.50,0.50) & (-1.00,1.00,0.00,0.00) \end{pmatrix}$$

$$M_4 = \begin{pmatrix} (-1.00,1.00,0.00,0.00) & (-0.50,0.50,-0.50,0.50) & (-0.33,0.33,-0.67,0.67) & (-0.33,0.33,-0.67,0.67) \\ (-0.50,0.50,-0.50,0.50) & (-1.00,1.00,0.00,0.00) & (-0.50,0.50,-0.50,0.50) & (0.00,0.00,-1.00,1.00) \\ (-0.33,0.33,-0.67,0.67) & (-0.50,0.50,-0.50,0.50) & (-1.00,1.00,0.00,0.00) & (-0.33,0.33,-0.67,0.67) \\ (-0.33,0.33,-0.67,0.67) & (0.00,0.00,-1.00,1.00) & (-0.33,0.33,-0.67,0.67) & (-1.00,1.00,0.00,0.00) \end{pmatrix}$$

The relation matrix $M_{\mathcal{Q}}$ is given by $S_{\mathcal{Q}}$, defined as:

$$M_{\mathcal{Q}} = \left(\bigvee_{k=1}^{4} \pi_{M_k}^{N}(u_i,u_j), \bigwedge_{k=1}^{4} \pi_{M_k}^{P}(u_i,u_j, \bigwedge_{k=1}^{4} \eta_{M_k}^{N}(u_i,u_j), \bigvee_{k=1}^{4} \eta_{M_k}^{P}(u_i,u_j)\right)$$

$$M_{\mathcal{Q}} = \begin{pmatrix} (-1.00,1.00,0.00,0.00) & (0.00,0.00,-1.00,1.00) & (-0.33,0.33,-0.67,0.67) & (0.00,0.00,-1.00,1.00) \\ (0.00,0.00,-1.00,1.00) & (-1.00,1.00,0.00,0.00) & (0.00,0.00,-1.00,1.00) & (0.00,0.00,-1.00,1.00) \\ (-0.33,0.33,-0.67,0.67) & (0.00,0.00,-1.00,1.00) & (-1.00,1.00,0.00,0.00) & (-0.33,0.33,-0.67,0.67) \\ (0.00,0.00,-1.00,1.00) & (0.00,0.00,-1.00,1.00) & (-0.33,0.33,-0.67,0.67) & (-1.00,1.00,0.00,0.00) \end{pmatrix}.$$

Accordingly, $(\mathcal{U}, \mathcal{Q})$ forms a $BIFRS$. The positive discernibility matrix $M^{+*} = (s_{ij}^{+*})_{4\times 4}$ is presented as follows:

$$M^{+*} = \begin{pmatrix} \emptyset & \emptyset & \{S_1,S_3,S_4\} & \emptyset \\ \emptyset & \emptyset & \emptyset & \emptyset \\ \{S_1,S_3,S_4\} & \mathcal{Q} & \emptyset & \{S_1,S_4\} \\ \emptyset & \emptyset & \{S_1,S_4\} & \emptyset \end{pmatrix}.$$

Similarly, the discernibility matrix $M^{-*} = (s_{ij}^{-*})_{4\times 4}$ is given by:

$$M^{-*} = \begin{pmatrix} \emptyset & \emptyset & \{S_1,S_3,S_4\} & \emptyset \\ \emptyset & \emptyset & \emptyset & \emptyset \\ \{S_1,S_3,S_4\} & \mathcal{Q} & \emptyset & \{S_1,S_4\} \\ \emptyset & \emptyset & \{S_1,S_4\} & \emptyset \end{pmatrix}.$$

Routine computation confirms that $M^{-*} = M^{+*}$, and the discernibility functions satisfy $h^+ = S_1 \wedge S_4 = h^-$. Consequently, both $O_1 = \{S_1\} \subset \mathcal{Q}$ and $O_2 = \{S_4\} \subset \mathcal{Q}$ are reducts of the $BIFRS(\mathcal{U}, \mathcal{Q})$. Moreover, it is straightforward to verify that $S_{O_1}(u_i,u_j) = S_{O_2}(u_i,u_j) = S_{\mathcal{Q}}(u_i,u_j)$ holds for all $u_i, u_j \in \mathcal{U}$.

Example 2 considered the extreme cases of hesitation degrees $\sigma^P = 0$ and $\sigma^N = 0$. Below, we introduce a tuning parameter to generate non-zero hesitation degrees. The tuning parameter $\gamma \in [0,1]$ is used as follows:

$$\sigma^P = \gamma \cdot (1 - \pi^P - \eta^P) + (1 - \gamma) \cdot \gamma,$$
$$\sigma^N = \gamma \cdot (-1 - \pi^N - \eta^N) - (1 - \gamma) \cdot \gamma.$$

By adjusting the value of γ, when $\gamma = 0$, $\sigma^P = 0$ and $\sigma^N = 0$ degenerate to the extreme no-hesitation case in Example 2. When $\gamma = 1$, $\sigma^P = 1 - \pi^P - \eta^P$ and $\sigma^N = -1 - \pi^N - \eta^N$ recover the definition of hesitation degrees in intuitionistic fuzzy sets ($\sigma = 1 - \pi - \eta$). This allows for a more nuanced description of uncertainty in the decision-making process, providing the model with greater flexibility to accurately simulate complex real-world scenarios.

Example 3

Based on the table from Example 2, let $\sigma^P = |\sigma^N|$, and $\gamma = 0.3$. The corresponding new relation matrices are shown as follows:

$$M_1 = \left(\begin{matrix} (-1.00, 1.00, 0.00, 0.00) & (0.00, 0.00, -0.79, 0.79) \\ (0.00, 0.00, -79.0, 0.79) & (-1.00, 1.00, 0.00, 0.00) \\ (-0.33, 0.33, -0.46, 0.46) & (-0.50, 0.50, -0.29, 0.29) \\ (-1.00, 1.00, 0.00, 0.00) & (0.00, 0.00, -0.79, 0.79) \end{matrix}\right.$$
$$\left.\begin{matrix} (-0.33, 0.33, -0.46, 0.46) & (-1.00, 1.00, 0.00, 0.00) \\ (-0.50, 0.50, -0.29, 0.29) & (0.00, 0.00, -0.79, 0.79) \\ (-1.00, 1.00, 0.00, 0.00) & (-0.33, 0.33, -0.46, 0.46) \\ (-0.33, 0.33, -0.46, 0.46) & (-1.00, 1.00, 0.00, 0.00) \end{matrix}\right)$$

$$M_2 = \left(\begin{matrix} (-1.00, 1.00, 0.00, 0.00) & (0.00, 0.00, -0.79, 0.79) \\ (0.00, 0.00, -0.79, 0.79) & (-1.00, 1.00, 0.00, 0.00) \\ (-0.50, 0.50, -0.29, 0.29) & (-0.50, 0.50, -0.29, 0.29) \\ (0.00, 0.00, -0.79, 0.79) & (-1.00, 1.00, 0.00, 0.00) \end{matrix}\right.$$
$$\left.\begin{matrix} (-0.50, 0.50, -0.29, 0.29) & (0.00, 0.00, -0.79, 0.79) \\ (-0.50, 0.50, -0.29, 0.29) & (-1.00, 1.00, 0.00, 0.00) \\ (-1.00, 1.00, 0.00, 0.00) & (-0.5, 0.5, -0.29, 0.29) \\ (-0.50, 0.50, -0.29, 0.29) & (-1.00, 1.00, 0.00, 0.00) \end{matrix}\right)$$

$$M_3 = \left(\begin{matrix} (-1.00, 1.00, 0.00, 0.00) & (-0.33, 0.33, -0.46, 0.46) \\ (-0.33, 0.33, -0.46, 0.46) & (-1.00, 1.00, 0.00, 0.00) \\ (-0.33, 0.33, -0.46, 0.46) & (0.00, 0.00, -0.79, 0.79) \\ (-0.67, 0.67, -0.21, 0.21) & (-0.50, 0.50, -0.29, 0.29) \end{matrix}\right.$$
$$\left.\begin{matrix} (-0.33, 0.33, -0.46, 0.46) & (-0.67, 0.67, -0.12, 0.12) \\ (0.00, 0.00, -0.79, 0.79) & (-0.50, 0.50, -0.29, 0.29) \\ (-1.00, 1.00, 0.00, 0.00) & (-0.50, 0.50, -0.29, 0.29) \\ (-0.50, 0.50, -0.29, 0.29) & (-1.00, 1.00, 0.00, 0.00) \end{matrix}\right)$$

$$M_4 = \left(\begin{matrix} (-1.00, 1.00, 0.00, 0.00) & (-0.50, 0.50, -0.29, 0.29) \\ (-0.50, 0.50, -0.29, 0.29) & (-1.00, 1.00, 0.00, 0.00) \\ (-0.33, 0.33, -0.46, 0.46) & (-0.50, 0.50, -0.29, 0.29) \\ (-0.33, 0.33, -0.46, 0.46) & (0.00, 0.00, -0.79, 0.79) \end{matrix}\right.$$
$$\left.\begin{matrix} (-0.33, 0.33, -0.46, 0.46) & (-0.33, 0.33, -0.46, 0.46) \\ (-0.50, 0.50, -0.29, 0.29) & (0.00, 0.00, -0.79, 0.79) \\ (-1.00, 1.00, 0.00, 0.00) & (-0.33, 0.33, -0.46, 0.46) \\ (-0.33, 0.33, -0.46, 0.46) & (-1.00, 1.00, 0.00, 0.00) \end{matrix}\right)$$

The relation matrix $M_{\mathcal{Q}}$ is given by $S_{\mathcal{Q}}$, defined as:

$$M_Q =$$
$$\left(\bigvee_{k=1}^{4} \pi^{N}_{M_k}(u_i, u_j), \bigwedge_{k=1}^{4} \pi^{P}_{M_k}(u_i, u_j, \bigvee_{k=1}^{4} \eta^{N}_{M_k}(u_i, u_j), \bigwedge_{k=1}^{4} \eta^{P}_{M_k}(u_i, u_j)\right)$$

$$M_{\mathcal{Q}} =$$
$$\left(\begin{matrix} (-1.00, 1.00, 0.00, 0.00) & (0.00, 0.00, -0.79, 0.79) \\ (0.00, 0.00, -0.79, 0.79) & (-1.00, 1.00, 0.00, 0.00) \\ (-0.33, 0.33, -0.46, 0.46) & (0.00, 0.00, -0.79, 0.79) \\ (0.00, 0.00, -0.79, 0.79) & (0.00, 0.00, -0.79, 0.79) \end{matrix}\right.$$
$$\left.\begin{matrix} (-0.33, 0.33, -0.46, 0.46) & (0.00, 0.00, -0.79, 0.79) \\ (0.00, 0.00, -0.79, 0.79) & (0.00, 0.00, -0.79, 0.79) \\ (-1.00, 1.00, 0.00, 0.00) & (-0.33, 0.33, -0.46, 0.46) \\ (-0.33, 0.33, -0.46, 0.46) & (-1.00, 1.00, 0.00, 0.00) \end{matrix}\right).$$

Accordingly, $(\mathcal{U}, \mathcal{Q})$ forms a $BIFRS$. The positive discernibility matrix $M^{+*} = (s^{+*}_{ij})_{4\times 4}$ is presented as follows:

$$M^{+*} = \begin{pmatrix} \emptyset & \{S_1, S_2\} & \{S_1, S_3, S_4\} & \{S_2\} \\ \{S_1, S_2\} & \emptyset & \{S_3\} & \{S_1, S_4\} \\ \{S_1, S_3, S_4\} & \{S_3\} & \emptyset & \{S_1, S_4\} \\ \{S_2\} & \{S_1, S_4\} & \{S_1, S_4\} & \emptyset \end{pmatrix}.$$

Likewise, the negative discernibility matrix $M^{-*} = (s^{-*}_{ij})_{4\times 4}$ is expressed as:

$$M^{-*} = \begin{pmatrix} \emptyset & \{S_1, S_2\} & \{S_1, S_3, S_4\} & \{S_2\} \\ \{S_1, S_2\} & \emptyset & \{S_3\} & \{S_1, S_4\} \\ \{S_1, S_3, S_4\} & \{S_3\} & \emptyset & \{S_1, S_4\} \\ \{S_2\} & \{S_1, S_4\} & \{S_1, S_4\} & \emptyset \end{pmatrix}.$$

Through routine calculations, it can be easily verified that $M^{-*} = M^{+*}$ and the discernibility functions $h^{+} = (S_1 \wedge S_4) \vee S_2 \vee S_3 = h^{-}$. Similarly, when $\gamma = 0.7$, the discernibility function yields $h^{+} = (S_1 \wedge S_4) \vee S_2 \vee S_3 = h^{-}$.

Through the verification of different values of γ, it has been observed that regardless of the value of γ, additional fixed reductions are obtained on the basis of the extreme cases with non-zero hesitation degrees $\sigma^P = 0$ and $\sigma^N = 0$ (refer to Examples 2 and 3). Moreover, the reductions under these extreme conditions remain the core reductions. In the application section that follows, only the extreme cases with non-zero hesitation degrees $\sigma^P = 0$ and $\sigma^N = 0$ were considered.

Illustrative Examples

Reduction of Myocardial Infarction Diagnostic Data

Table 2 presents a set-valued information system that describes the diagnosis of myocardial infarction in six patients. The set of patients is denoted by $\mathcal{U} = \{u_1, u_2, \ldots, u_6\}$, and the set of enzymes relevant to the diagnosis is $\mathcal{Q} = \{S_1, S_2, S_3, S_4, S_5\}$. Here, S_1 corresponds to aspartate aminotransferase, S_2 refers to lactate dehydrogenase and its isoenzymes, S_3 represents alfa-hydroxybutyrate dehydrogenase, S_4 stands for creatine kinase, and S_5 indicates creatine kinase isoenzymes. The primary aim is to simplify diagnostic attributes by converting the set-valued information system into a bipolar intuitionistic fuzzy discernibility matrix.

Table 2: A set-valued information table regarding myocardial infarction in patients

U	S_1	S_2	S_3	S_4	S_5
u_1	$\{1, 3\}$	$\{1\}$	$\{2\}$	$\{1, 2, 3\}$	$\{3\}$
u_2	$\{2\}$	$\{1; 3\}$	$\{1, 2\}$	$\{2, 3\}$	$\{2\}$
u_3	$\{1\}$	$\{1, 3\}$	$\{2, 3\}$	$\{1\}$	$\{3\}$
u_4	$\{1, 2\}$	$\{2\}$	$\{1, 2, 3\}$	$\{1\}$	$\{1, 2\}$
u_5	$\{2, 3\}$	$\{2\}$	$\{3\}$	$\{2, 3\}$	$\{2\}$
u_6	$\{2, 3\}$	$\{1, 2\}$	$\{2, 3\}$	$\{1\}$	$\{1, 2, 3\}$

Using Algorithm 2, each can be calculated as a BIF relation over. The corresponding relation matrix defines each BIF relation S_k, as follows:

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$$M_1 = \left(\begin{matrix} (-1.00, 1.00, 0.00, 0.00) & (0.00, 0.00, -1.00, 1.00) & (-0.50, 0.50, -0.50, 0.50) \\ (0.00, 0.00, -1.00, 1.00) & (-1.00, 1.00, 0.00, 0.00) & (0.00, 0.00, -1.00, 1.00) \\ (-0.50, 0.50, -0.50, 0.50) & (0.00, 0.00, -1.00, 1.00) & (-1.00, 1.00, 0.00, 0.00) \\ (-0.33, 0.33, -0.67, 0.67) & (-0.50, 0.50, -0.50, 0.50) & (-0.50, 0.50, -0.50, 0.50) \\ (-0.33, 0.33, -0.67, 0.67) & (-0.50, 0.50, -0.50, 0.50) & (0.00, 0.00, -1.00, 1.00) \\ (-0.33, 0.33, -0.67, 0.67) & (-0.50, 0.50, -0.50, 0.50) & (0.00, 0.00, -1.00, 1.00) \end{matrix} \right.$$

$$\left. \begin{matrix} (-0.33, 0.33, -0.67, 0.67) & (-0.33, 0.33, -0.67, 0.67) & (-0.33, 0.33, -0.67, 0.67) \\ (-0.50, 0.50, -0.50, 0.50) & (-0.50, 0.50, -0.50, 0.50) & (-0.50, 0.50, -0.50, 0.50) \\ (-0.50, 0.50, -0.50, 0.50) & (0.00, 0.00, -1.00, 1.00) & (0.00, 0.00, -1.00, 1.00) \\ (-1.00, 1.00, 0.00, 0.00) & (-0.33, 0.33, -0.67, 0.67) & (-0.33, 0.33, -0.67, 0.67) \\ (-0.33, 0.33, -0.67, 0.67) & (-1.00, 1.00, 0.00, 0.00) & (-1.00, 1.00, 0.00, 0.00) \\ (-0.33, 0.33, -0.67, 0.67) & (-1.00, 1.00, 0.00, 0.00) & (-1.00, 1.00, 0.00, 0.00) \end{matrix} \right)$$

$$M_2 = \left(\begin{matrix} (-1.00, 1.00, 0.00, 0.00) & (-0.50, 0.50, -0.50, 0.50) & (-1.00, 1.00, 0.00, 0.00) \\ (-0.50, 0.50, -0.50, 0.50) & (-1.00, 1.00, 0.00, 0.00) & (-0.50, 0.50, -0.50, 0.50) \\ (-1.00, 1.00, 0.00, 0.00) & (-0.50, 0.50, -0.50, 0.50) & (-1.00, 1.00, 0.00, 0.00) \\ (0.00, 0.00, -1.00, 1.00) & (0.00, 0.00, -1.00, 1.00) & (0.00, 0.00, -1.00, 1.00) \\ (0.00, 0.00, -1.00, 1.00) & (0.00, 0.00, -1.00, 1.00) & (0.00, 0.00, -1.00, 1.00) \\ (-0.50, 0.50, -0.50, 0.50) & (-0.33, 0.33, -0.67, 0.67) & (-0.50, 0.50, -0.50, 0.50) \end{matrix} \right.$$

$$\left. \begin{matrix} (0.00, 0.00, -1.00, 1.00) & (0.00, 0.00, -1.00, 1.00) & (-0.50, 0.50, -0.50, 0.50) \\ (0.00, 0.00, -1.00, 1.00) & (0.00, 0.00, -1.00, 1.00) & (-0.33, 0.33, -0.67, 0.67) \\ (0.00, 0.00, -1.00, 1.00) & (0.00, 0.00, -1.00, 1.00) & (-0.50, 0.50, -0.50, 0.50) \\ (-1.00, 1.00, 0.00, 0.00) & (-1.0, 01.00, 0.00, 0.00) & (-0.50, 0.50, -0.50, 0.50) \\ (-1.00, 1.00, 0.00, 0.00) & (-1.00, 1.00, 0.00, 0.00) & (-0.50, 0.50, -0.50, 0.50) \\ (-0.50, 0.50, -0.50, 0.50) & (-0.50, 0.50, -0.50, 0.50) & (-1.00, 1.00, 0.00, 0.00) \end{matrix} \right)$$

$$M_3 = \left(\begin{matrix} (-1.00, 1.00, 0.00, 0.00) & (-0.50, 0.50, -0.50, 0.50) & (-0.50, 0.50, -0.50, 0.50) \\ (-0.50, 0.50, -0.50, 0.50) & (-1.00, 1.00, 0.00, 0.00) & (-0.33, 0.33, -0.67, 0.67) \\ (-0.50, 0.50, -0.50, 0.50) & (-0.33, 0.33, -0.67, 0.67) & (-1.00, 1.00, 0.00, 0.00) \\ (-0.33, 0.33, -0.67, 0.67) & (-0.67, 0.67, -0.33, 0.33) & (-0.67, 0.67, -0.33, 0.33) \\ (0.00, 0.00, -1.00, 1.00) & (0.00, 0.00, -1.00, 1.00) & (-0.50, 0.50, -0.50, 0.50) \\ (-0.50, 0.50, -0.50, 0.50) & (-0.33, 0.33, -0.67, 0.67) & (-1.00, 1.00, 0.00, 0.00) \end{matrix} \right.$$

$$\left. \begin{matrix} (-0.33, 0.33, -0.67, 0.67) & (0.00, 0.00, -1.00, 1.00) & (-0.50, 0.50, -0.50, 0.50) \\ (-0.67, 0.67, -0.33, 0.33) & (0.00, 0.00, -1.00, 1.00) & (-0.33, 0.33, -0.67, 0.67) \\ (-0.67, 0.67, -0.33, 0.33) & (-0.50, 0.50, -0.50, 0.50) & (-1.00, 1.00, 0.00, 0.00) \\ (-1.00, 1.00, 0.00, 0.00) & (-0.33, 0.33, -0.67, 0.67) & (-0.67, 0.67, -0.33, 0.33) \\ (-0.33, 0.33, -0.67, 0.67) & (-1.00, 1.00, 0.00, 0.00) & (-0.50, 0.50, -0.50, 0.50) \\ (-0.67, 0.67, -0.33, 0.33) & (-0.50, 0.50, -0.50, 0.50) & (-1.00, 1.00, 0.00, 0.00) \end{matrix} \right)$$

$$M_4 = \left(\begin{matrix} (-1.00, 1.00, 0.00, 0.00) & (-0.67, 0.67, -0.33, 0.33) & (-0.33, 0.33, -0.67, 0.67) \\ (-0.67, 0.67, -0.33, 0.33) & (-1.00, 1.00, 0.00, 0.00) & (0.00, 0.00, -1.00, 1.00) \\ (-0.33, 0.33, -0.67, 0.67) & (0.00, 0.00, -1.00, 1.00) & (-1.00, 1.00, 0.00, 0.00) \\ (-0.33, 0.33, -0.67, 0.67) & (0.00, 0.00, -1.00, 1.00) & (-1.00, 1.00, 0.00, 0.00) \\ (-0.67, 0.67, -0.33, 0.33) & (-1.00, 1.00, 0.00, 0.00) & (0.00, 0.00, -1.00, 1.00) \\ (-0.33, 0.33, -0.67, 0.67) & (0.00, 0.00, -1.00, 1.00) & (-1.00, 1.00, 0.00, 0.00) \end{matrix} \right.$$

$$\left. \begin{matrix} (-0.33, 0.33, -0.67, 0.67) & (-0.67, 0.67, -0.33, 0.33) & (-0.33, 0.33, -0.67, 0.67) \\ (0.00, 0.00, -1.00, 1.00) & (-1.00, 1.00, 0.00, 0.00) & (0.00, 0.00, -1.00, 1.00) \\ (-1.00, 1.00, 0.00, 0.00) & (0.00, 0.00, -1.00, 1.00) & (-1.00, 1.00, 0.00, 0.00) \\ (-1.00, 1.00, 0.00, 0.00) & (0.00, 0.00, -1.00, 1.00) & (-1.00, 1.00, 0.00, 0.00) \\ (0.00, 0.00, -1.00, 1.00) & (-1.00, 1.00, 0.00, 0.00) & (0.00, 0.0, -1.00, 1.00) \\ (-1.00, 1.00, 0.00, 0.00) & (0.00, 0.00, -1.00, 1.00) & (-1.00, 1.00, 0.00, 0.00) \end{matrix} \right)$$

$$M_5 = \begin{pmatrix} (-1.00, 1.00, 0.00, 0.00) & (0.00, 0.00, -1.00, 1.00) & (-1.00, 1.00, 0.00, 0.00) \\ (0.00, 0.00, -1.00, 1.00) & (-1.00, 1.00, 0.00, 0.00) & (0.00, 0.00, -1.00, 1.00) \\ (-1.00, 1.00, 0.00, 0.00) & (0.00, 0.00, -1.00, 1.00) & (-1.00, 1.0, 0.00, 0.00) \\ (0.00, 0.00, -1.00, 1.00) & (-0.50, 0.50, -0.50, 0.50) & (0.00, 0.00, -1.00, 1.00) \\ (0.00, 0.00, -1.00, 1.00) & (-1.00, 1.00, 0.00, 0.00) & (0.00, 0.00, -1.00, 1.00) \\ (-0.33, 0.33, -0.67, 0.67) & (-0.33, 0.33, -0.67, 0.67) & (-0.33, 0.33, -0.67, 0.67) \end{matrix}$$
$$\begin{matrix} (0.0, 0.0, -1.0, 1.0) & (0.0, 0.0, -1.0, 1.0) & (-0.33, 0.33, -0.67, 0.67) \\ (-0.5, 0.5, -0.5, 0.5) & (-1.0, 1.0, 0.0, 0.0) & (-0.33, 0.33, -0.67, 0.67) \\ (0.0, 0.0, -1.0, 1.0) & (0.0, 0.0, -1.0, 1.0) & (-0.33, 0.33, -0.67, 0.67) \\ (-1.0, 1.0, 0.0, 0.0) & (-0.5, 0.5, -0.5, 0.5) & (-0.67, 0.67, -0.33, 0.33) \\ (-0.5, 0.5, -0.5, 0.5) & (-1.0, 1.0, 0.0, 0.0) & (-0.33, 0.33, -0.67, 0.67) \\ (-0.67, 0.67, -0.33, 0.33) & (-0.33, 0.33, -0.67, 0.67) & (-1.0, 1.0, 0.0, 0.0) \end{pmatrix}$$

The discernibility matrix $M_{\mathcal{Q}}$ is given by $S_{\mathcal{Q}}$, defined as:

$$M_{\mathcal{Q}} = \left(\bigvee_{k=1}^{6} \pi_{M_k}^{N}(u_i, u_j), \bigwedge_{k=1}^{6} \pi_{M_k}^{P}(u_i, u_j), \bigwedge_{k=1}^{6} \eta_{M_k}^{N}(u_i, u_j), \bigvee_{k=1}^{6} \eta_{M_k}^{P}(u_i, u_j)\right)$$

$$M_{\mathcal{Q}} = \begin{pmatrix} (-1.00, 1.00, 0.00, 0.00) & (0.00, 0.00, -1.00, 1.00) & (-0.33, 0.33, -0.67, 0.67) \\ (0.00, 0.00, -1.00, 1.00) & (-1.00, 1.00, 0.00, 0.00) & (0.00, 0.00, -1.00, 1.00) \\ (-0.33, 0.33, -0.67, 0.67) & (0.00, 0.00, -1.00, 1.00) & (-1.00, 1.00, 0.00, 0.00) \\ (0.00, 0.00, -1.00, 1.00) & (0.00, 0.00, -1.00, 1.00) & (0.00, 0.00, -1.00, 1.00) \\ (0.00, 0.00, -1.00, 1.00) & (0.00, 0.00, -1.00, 1.00) & (0.00, 0.00, -1.00, 1.00) \\ (-0.33, 0.33, -0.67, 0.67) & (0.00, 0.00, -1.00, 1.00) & (0.00, 0.00, -1.00, 1.00) \end{matrix}$$
$$\begin{matrix} (0.00, 0.00, -1.00, 1.00) & (0.00, 0.00, -1.00, 1.00) & (-0.33, 0.33, -0.67, 0.67) \\ (0.00, 0.00, -1.00, 1.00) & (0.00, 0.00, -1.00, 1.00) & (0.00, 0.00, -1.00, 1.00) \\ (0.00, 0.00, -1.00, 1.00) & (0.00, 0.00, -1.00, 1.00) & (0.00, 0.00, -1.00, 1.00) \\ (-1.00, 1.00, 0.00, 0.00) & (0.00, 0.00, -1.00, 1.00) & (-0.33, 0.33, -0.67, 0.67) \\ (0.00, 0.00, -1.00, 1.00) & (-1.00, 1.00, 0.00, 0.00) & (0.00, 0.00, -1.00, 1.00) \\ (-0.33, 0.33, -0.67, 0.67) & (0.00, 0.00, -1.00, 1.00) & (-1.00, 1.00, 0.00, 0.00) \end{pmatrix}$$

Therefore, $(\mathcal{U}, \mathcal{Q})$ is a $BIFIS$. The discernibility matrix $M^{+*} = (s_{ij}^{+*})_{6\times 6}$ is provided as follows:

$$M^{+*} = \begin{pmatrix} \emptyset & \emptyset & \{S_4\} & \emptyset & \emptyset & \{S_1, S_4, S_5\} \\ \emptyset & \emptyset & \emptyset & \emptyset & \emptyset & \emptyset \\ \{S_4\} & \emptyset & \emptyset & \emptyset & \emptyset & \emptyset \\ \emptyset & \emptyset & \emptyset & \emptyset & \emptyset & \{S_1\} \\ \emptyset & \emptyset & \emptyset & \emptyset & \emptyset & \emptyset \\ \{S_1, S_4, S_5\} & \emptyset & \emptyset & \{S_1\} & \emptyset & \emptyset \end{pmatrix}.$$

Similarly, the discernibility matrix $M^{-*} = (s_{ij}^{-*})_{6\times 6}$ is provided as follows:

$$M^{-*} = \begin{pmatrix} \emptyset & \emptyset & \{S_4\} & \emptyset & \emptyset & \{S_1, S_4, S_5\} \\ \emptyset & \emptyset & \emptyset & \emptyset & \emptyset & \emptyset \\ \{S_4\} & \emptyset & \emptyset & \emptyset & \emptyset & \emptyset \\ \emptyset & \emptyset & \emptyset & \emptyset & \emptyset & \{S_1\} \\ \emptyset & \emptyset & \emptyset & \emptyset & \emptyset & \emptyset \\ \{S_1, S_4, S_5\} & \emptyset & \emptyset & \{S_1\} & \emptyset & \emptyset \end{pmatrix}.$$

Routine calculations readily confirm that $M^{-*} = M^{+*}$, and for the discernibility function, $h^{+} = S_1 \wedge S_4 = h^{-}$. It follows that $O_1 = \{S_1\}$ and $O_2 = \{S_4\}$ serve as the reduction sets for $BIFRS(\mathcal{U}, \mathcal{Q})$. This is evident since $S_{O_1}(u_i, u_j) = S_{O_2}(u_i, u_j) = S_{\mathcal{Q}}(u_i, u_j)$ holds true for all $u_i, u_j \in \mathcal{U}$. Clearly, the sets $\{S_1\}$ and $\{S_4\}$ constitute the cores of the reduction sets.

Risk Investment Data Reduction

Investing in business typically involves a variety of risks, with two main categories being systematic and unsystematic risks prevalent in economic markets. Systematic risk, also referred to as market risk, involves risks tied to the broader economy or investment markets. Unsystematic risk, on the other hand, is typically associated with individual companies or specific investments. The characteristics of these risks play a crucial role in shaping how investors make decisions. For instance, consider a venture capital firm evaluating six potential investment projects, denoted as $\mathcal{U} = \{u_1, u_2, \ldots, u_6\}$. These projects are assessed according to multiple risk factors, such as inflation risk, technological risk, operational risk, political risk, and market risk. These criteria are designed to reflect preferences, and expert evaluations using interval numbers rate each project's performance under each factor. The decision-making table provided by experts (Table 3) outlines the risk profiles of various company projects. The set of risks faced by the company is represented as $\mathcal{Q} = \{S_1, S_2, S_3, S_4, S_5\}$, corresponding respectively to inflation risk, technological risk, operational risk, political risk, and market risk. The varying intervals of decision attributes indicate diverse

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risk outcomes. The primary aim of this study is to streamline the analysis of risk attributes by transforming the interval-valued information system into a bipolar intuitionistic fuzzy discernibility matrix. This transformation aims to streamline the evaluation process and enhance the clarity of risk assessment across the considered projects.

Utilizing Algorithm 2, each $S_k, (k = 1, 2, 3, 4, 5)$ can be expressed as a BIF relation over $\mathcal{U}$. The corresponding relational matrices for each S_k are presented as follows:

Table 3: A risk investment interval-valued information table for a company

U	S_1	S_2	S_3	S_4	S_5
u_1	[3.12, 4.56]	[5.76, 6.64]	[7.92, 9.21]	[1.14, 3.21]	[8.27, 10.13]
u_2	[4.07, 5.18]	[6.31, 7.20]	[8.01, 9.37]	[1.75, 3.86]	[9.08, 10.49]
u_3	[4.26, 5.37]	[5.03, 5.91]	[7.87, 9.23]	[1.64, 3.75]	[7.40, 8.52]
u_4	[3.00, 4.44]	[5.83, 6.71]	[7.01, 8.21]	[1.20, 3.30]	[7.85, 9.71]
u_5	[4.31, 6.14]	[2.61, 4.32]	[5.12, 10.22]	[1.34, 4.91]	[5.26, 7.61]
u_6	[3.11, 6.61]	[3.33, 5.90]	[2.81, 6.31]	[2.22, 4.02]	[5.44, 8.74]

$$M_1 = \left(\begin{matrix} (-1.00, 1.00, 0.00, 0.00) & (0.24, 0.24, -0.76, 0.76) & (-0.13, 0.13, -0.87, 0.87) \\ (0.24, 0.24, -0.76, 0.76) & (-1.00, 1.00, 0.00, 0.00) & (-0.71, 0.71, -0.29, 0.29) \\ (-0.13, 0.13, -0.87, 0.87) & (-0.71, 0.71, -0.29, 0.29) & (-1.00, 1.00, 0.00, 0.00) \\ (-0.85, 0.85, -0.15, 0.15) & (-0.08, 0.08, -0.92, 0.92) & (-0.08, 0.08, -0.92, 0.92) \\ (-0.08, 0.08, -0.92, 0.92) & (-0.42, 0.42, -0.58, 0.58) & (-0.61, 0.61, -0.39, 0.39) \\ (-0.40, 0.40, -0.60, 0.60) & (-0.32, 0.32, -0.68, 0.68) & (-0.32, 0.32, -0.68, 0.68) \end{matrix}\right.$$
$$\left.\begin{matrix} (-0.85, 0.85, -0.15, 0.15) & (-0.08, 0.08, -0.92, 0.92) & (-0.40, 0.40, -0.60, 0.60) \\ (-0.08, 0.08, -0.92, 0.92) & (-0.42, 0.42, -0.58, 0.58) & (-0.32, 0.32, -0.68, 0.68) \\ (-0.08, 0.08, -0.92, 0.92) & (-0.61, 0.61, -0.39, 0.39) & (-0.32, 0.32, -0.68, 0.68) \\ (-1.00, 1.00, 0.00, 0.00) & (-0.04, 0.04, -0.96, 0.96) & (-0.37, 0.37, -0.63, 0.63) \\ (-0.04, 0.04, -0.96, 0.96) & (-1.00, 1.00, 0.00, 0.00) & (-0.52, 0.52, -0.48, 0.48) \\ (-0.37, 0.37, -0.63, 0.63) & (-0.52, 0.52, -0.48, 0.48) & (-1.00, 1.00, 0.00, 0.00) \end{matrix}\right)$$

$$M_2 = \left(\begin{matrix} (-1.00, 1.00, 0.00, 0.00) & (-0.23, 0.23, -0.77, 0.77) & (-0.09, 0.09, -0.91, 0.91) \\ (-0.23, 0.23, -0.77, 0.77) & (-1.00, 1.00, 0.00, 0.00) & (0.00, 0.00, -1.00, 1.00) \\ (-0.09, 0.09, -0.91, 0.91) & (0.00, 0.00, -1.00, 1.00) & (-1.00, 1.00, 0.00, 0.00) \\ (-0.85, 0.85, -0.15, 0.15) & (-0.30, 0.30, -0.70, 0.70) & (-0.05, 0.05, -0.95, 0.95) \\ (0.00, 0.00, -1.00, 1.00) & (0.00, 0.00, -1.00, 1.00) & (0.00, 0.00, -1.00, 1.00) \\ (-0.04, 0.04, -0.96, 0.96) & (0.00, 0.00, -1.00, 1.00) & (-0.34, 0.34, -0.66, 0.66) \end{matrix}\right.$$
$$\left.\begin{matrix} (-0.85, 0.85, -0.15, 0.15) & (0.00, 0.00, -1.00, 1.00) & (-0.04, 0.04, -0.96, 0.96) \\ (-0.30, 0.30, -0.70, 0.70) & (0.00, 0.00, -1.00, 1.00) & (0.00, 0.00, -1.00, 1.00) \\ (-0.05, 0.05, -0.95, 0.95) & (0.00, 0.00, -1.00, 1.00) & (-0.34, 0.34, -0.66, 0.66) \\ (-1.00, 1.00, 0.00, 0.00) & (0.00, 0.00, -1.00, 1.00) & (-0.02, 0.02, -0.98, 0.98) \\ (0.00, 0.00, -1.00, 1.00) & (-1.00, 1.00, 0.00, 0.00) & (-0.30, 0.3, -0.70, 0.70) \\ (-0.02, 0.02, -0.98, 0.98) & (-0.30, 0.30, -0.70, 0.70) & (-1.00, 1.00, 0.00, 0.00) \end{matrix}\right)$$

$$M_3 = \left(\begin{matrix} (-1.00, 1.00, 0.00, 0.00) & (-0.83, 0.83, -0.17, 0.17) & (-0.85, 0.85, -0.15, 0.15) \\ (-0.83, 0.83, -0.17, 0.17) & (-1.00, 1.00, 0.00, 0.00) & (-0.81, 0.81, -0.19, 0.19) \\ (-0.85, 0.85, -0.15, 0.15) & (-0.81, 0.81, -0.19, 0.19) & (-1.00, 1.00, 0.00, 0.00) \\ (-0.13, 0.13, -0.87, 0.87) & (-0.08, 0.08, -0.92, 0.92) & (-0.15, 0.15, -0.85, 0.85) \\ (-0.25, 0.25, -0.75, 0.75) & (-0.27, 0.27, -0.73, 0.73) & (-0.27, 0.27, -0.73, 0.73) \\ (0.00, 0.00, -1.00, 1.00) & (0.00, 0.00, -1.00, 1.00) & (0.00, 0.00, -1.00, 1.00) \end{matrix}\right.$$
$$\left.\begin{matrix} (-0.13, 0.13, -0.87, 0.87) & (-0.25, 0.25, -0.75, 0.75) & (0.00, 0.00, -1.00, 1.00) \\ (-0.08, 0.08, -0.92, 0.92) & (-0.27, 0.27, -0.73, 0.73) & (0.00, 0.00, -1.00, 1.00) \\ (-0.15, 0.15, -0.85, 0.85) & (-0.27, 0.27, -0.73, 0.73) & (0.00, 0.00, -1.00, 1.00) \\ (-1.00, 1.00, 0.00, 0.00) & (-0.24, 0.24, -0.76, 0.76) & (0.00, 0.00, -1.00, 1.00) \\ (-0.24, 0.24, -0.76, 0.76) & (-1.0, 1.0, 0.0, 0.0) & (-016, 0.16, -0.84, 0.84) \\ (0.00, 0.00, -1.00, 1.00) & (-016, 0.16, -0.84, 0.84) & (-1.00, 1.00, 0.00, 0.00) \end{matrix}\right)$$

$$M_4 = \begin{pmatrix} (-1.00, 1.00, 0.00, 0.00) & (-0.54, 0.54, -0.46, 0.46) & (-0.60, 0.60, -0.40, 0.40) & (-0.93, 0.93, -0.07, 0.07) & (-0.50, 0.50, -0.50, 0.50) & (-0.34, 0.34, -0.66, 0.66) \\ (-0.54, 0.54, -0.46, 0.46) & (-1.00, 1.00, 0.00, 0.00) & (-0.89, 0.89, -0.11, 0.11) & (-0.58, 0.58, -0.42, 0.42) & (-0.59, 0.59, -0.41, 0.41) & (-0.72, 0.72, -0.28, 0.28) \\ (-0.60, 0.60, -0.40, 0.40) & (-0.89, 0.89, -0.11, 0.11) & (-1.00, 1.00, 0.00, 0.00) & (-0.65, 0.65, -0.35, 0.35) & (-0.59, 0.59, -0.41, 0.41) & (-0.64, 0.64, -0.36, 0.36) \\ (-0.93, 0.93, -0.07, 0.07) & (-0.58, 0.58, -0.42, 0.42) & (-0.65, 0.65, -0.35, 0.35) & (-1.00, 1.00, 0.00, 0.00) & (-0.53, 0.53, -0.47, 0.47) & (-0.38, 0.38, -0.62, 0.62) \\ (-0.50, 0.50, -0.50, 0.50) & (-0.59, 0.59, -0.41, 0.41) & (-0.59, 0.59, -0.41, 0.41) & (-0.53, 0.53, -0.47, 0.47) & (-1.00, 1.00, 0.00, 0.00) & (-0.50, 0.50, -0.50, 0.50) \\ (-0.34, 0.34, -0.66, 0.66) & (-0.72, 0.72, -0.28, 0.28) & (-0.64, 0.64, -0.36, 0.36) & (-0.38, 0.38, -0.62, 0.62) & (-0.50, 0.50, -0.50, 0.50) & (-1.00, 1.00, 0.00, 0.00) \end{pmatrix}$$

$$M_5 = \begin{pmatrix} (-1.00, 1.00, 0.00, 0.00) & (-0.47, 0.47, -0.53, 0.53) & (-0.09, 0.09, -0.91, 0.91) & (-0.63, 0.63, -0.37, 0.37) & (0.00, 0.00, -1.00, 1.00) & (-0.10, 0.10, -0.90, 0.90) \\ (-0.47, 0.47, -0.53, 0.53) & (-1.00, 1.00, 0.00, 0.00) & (0.00, 0.00, -1.00, 1.00) & (-0.24, 0.24, -0.76, 0.76) & (0.00, 0.00, -1.00, 1.00) & (0.00, 0.00, -1.00, 1.00) \\ (-0.09, 0.09, -0.91, 0.91) & (0.00, 0.00, -1.00, 1.00) & (-1.00, 1.00, 0.00, 0.00) & (-0.29, 0.29, -0.71, 0.71) & (-0.06, 0.06, -0.94, 0.94) & (-0.34, 0.34, -0.66, 0.66) \\ (-0.63, 0.63, -0.37, 0.37) & (-0.24, 0.24, -0.76, 0.76) & (-0.29, 0.29, -0.71, 0.71) & (-1.00, 1.00, 0.00, 0.00) & (0.00, 0.00, -1.00, 1.00) & (-0.21, 0.21, -0.79, 0.79) \\ (0.00, 0.00, -1.00, 1.00) & (0.00, 0.00, -1.00, 1.00) & (-0.06, 0.06, -0.94, 0.94) & (0.00, 0.00, -1.00, 1.00) & (-1.00, 1.00, 0.00, 0.00) & (-0.62, 0.62, -0.38, 0.38) \\ (-0.10, 0.10, -0.90, 0.90) & (0.00, 0.00, -1.00, 1.00) & (-0.34, 0.34, -0.66, 0.66) & (-0.21, 0.21, -0.79, 0.79) & (-0.62, 0.62, -0.38, 0.38) & (-1.00, 1.00, 0.00, 0.00) \end{pmatrix}$$

The discernibility matrix $M_{\mathcal{Q}}$ is given by $S_{\mathcal{Q}}$, defined as:

$$M_{\mathcal{Q}} = \left(\bigvee_{k=1}^{6} \pi^N_{M_k}(u_i, u_j), \bigwedge_{k=1}^{6} \pi^P_{M_k}(u_i, u_j), \bigwedge_{k=1}^{6} \eta^N_{M_k}(u_i, u_j), \bigvee_{k=1}^{6} \eta^P_{M_k}(u_i, u_j)\right) M_{\mathcal{Q}} =$$

$$\begin{pmatrix} (-1.00, 1.00, 0.00, 0.00) & (-0.23, 0.23, -0.77, 0.77) & (-0.09, 0.09, -0.91, 0.91) & (-0.13, 0.13, -0.87, 0.87) & (0.00, 0.00, -1.00, 1.00) & (0.00, 0.00, -1.00, 1.00) \\ (-0.23, 0.23, -0.77, 0.77) & (-1.00, 1.00, 0.00, 0.00) & (0.00, 0.00, -1.0, 1.00) & (-0.08, 0.08, -0.92, 0.92) & (0.00, 0.00, -1.00, 1.00) & (0.00, 0.00, -1.00, 1.00) \\ (-0.09, 0.09, -0.91, 0.91) & (0.00, 0.00, -1.00, 1.00) & (-1.00, 1.00, 0.00, 0.00) & (-0.05, 0.05, -0.95, 0.95) & (0.00, 0.00, -1.00, 1.00) & (0.00, 0.00, -1.00, 1.00) \\ (-0.13, 0.13, -0.87, 0.87) & (-0.08, 0.08, -0.92, 0.92) & (-0.05, 0.05, -0.95, 0.95) & (-1.00, 1.00, 0.00, 0.00) & (0.00, 0.00, -1.00, 1.00) & (0.00, 0.00, -1.00, 1.00) \\ (0.00, 0.00, -1.00, 1.00) & (0.00, 0.00, -1.00, 1.00) & (0.00, 0.00, -1.00, 1.00) & (0.00, 0.00, -1.00, 1.00) & (-1.00, 1.00, 0.00, 0.00) & (-0.16, 0.16, -0.84, 0.84) \\ (0.00, 0.00, -1.00, 1.00) & (0.00, 0.00, -1.00, 1.00) & (0.00, 0.00, -1.00, 1.00) & (0.00, 0.00, -1.00, 1.00) & (-0.16, 0.16, -0.84, 0.84) & (-1.00, 1.00, 0.00, 0.00) \end{pmatrix}$$

Accordingly, $(\mathcal{U}, \mathcal{Q})$ forms a $BIFRS$. The positive discernibility matrix $M^{+*} = (s^{+*}_{ij})_{6\times 6}$ is given as follows:

$$M^{+*} = \begin{pmatrix} \emptyset & \{S_2\} & \{S_2, S_5\} & \{S_3\} & \emptyset & \emptyset \\ \{S_2\} & \emptyset & \emptyset & \{S_1, S_3\} & \emptyset & \emptyset \\ \{S_2, S_5\} & \emptyset & \emptyset & \{S_2\} & \emptyset & \emptyset \\ \{S_3\} & \{S_1, S_3\} & \{S_2\} & \emptyset & \emptyset & \emptyset \\ \emptyset & \emptyset & \emptyset & \emptyset & \emptyset & \{S_3\} \\ \emptyset & \emptyset & \emptyset & \emptyset & \{S_3\} & \emptyset \end{pmatrix}.$$

Likewise, the negative discernibility matrix $M^{-*} = (s^{-*}_{ij})_{6\times 6}$ is given by:

$$M^{-*} = \begin{pmatrix} \emptyset & \{S_2\} & \{S_2, S_5\} & \{S_3\} & \emptyset & \emptyset \\ \{S_2\} & \emptyset & \emptyset & \{S_1, S_3\} & \emptyset & \emptyset \\ \{S_2, S_5\} & \emptyset & \emptyset & \{S_2\} & \emptyset & \emptyset \\ \{S_3\} & \{S_1, S_3\} & \{S_2\} & \emptyset & \emptyset & \emptyset \\ \emptyset & \emptyset & \emptyset & \emptyset & \emptyset & \{S_3\} \\ \emptyset & \emptyset & \emptyset & \emptyset & \{S_3\} & \emptyset \end{pmatrix}.$$

Standard computations verify that $M^{-*} = M^{+*}$, and the discernibility function satisfies $h^+ = S_2 \vee S_3 = h^-$. Consequently, $O_1 = \{S_2\}$ and $O_2 = \{S_3\}$ function as reduction sets for $BIFRS(\mathcal{U}, \mathcal{Q})$, since it is readily seen that $S_{O_1}(u_i, u_j) = S_{O_2}(u_i, u_j) = S_{\mathcal{Q}}(u_i, u_j)$ holds for all

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$u_i, u_j \in \mathcal{U}$. Evidently, the sets $\{S_2\}$ and $\{S_3\}$ form the cores of the reduction sets.

Reduction Operating on Large-Scale Datasets

lists the UCI public datasets used in this experiment. All datasets in the table were preprocessed by removing all instances with missing values. Each element in the datasets was converted into a bipolar intuitionistic fuzzy set. Detailed information is shown in Table 4.

Table 4: UCI and Related Datasets

No.	Dataset	$\|U\|$	$\|A\|$
1	Wine	178	13
2	Image Segmentation	210	19
3	Traffic Pedestrians	2061	14
4	Handwritten Digits	5620	64

In Table 4, $|U|$ and $|A|$ denote the sizes of the universe and the attribute set, respectively, for the given bipolar intuitionistic fuzzy information system. Attribute reduction algorithms based on a discernibility matrix typically operate in two phases: (1) construction of the discernibility matrix, and (2) computation of the reduct. The following analysis compares the execution times of the algorithms in both phases across each dataset.

Figure 1 shows the time proportions spent on two core steps of the BIFIS algorithm. The blue segments represent the proportion of time required for the first step (matrix generation), while the red segments indicate the time proportion for the second step (reduction calculation). As shown in Figure 1, the BIFIS algorithm consumes significantly more time on the first step across all datasets.The construction of discernibility matrices dominates the total runtime, with the matrix generation phase accounting for an average of 63.2% of execution time - significantly exceeding the reduction computation stage (36.8%). The time proportion allocated to generating these matrices exhibits a positive correlation with the number of attributes. For instance, in the Handwritten Digits dataset (with 64 attributes), this phase consumes up to 75% of the of the total time. The computational load demonstrates quadratic growth relative to attribute count (complexity $O(mn^2)$.

Figure 2 consists of two subfigures: the left subfigure compares the attribute counts of four UCI datasets before and after reduction using the BIFIS algorithm, where light blue bars represent the original attribute counts and red bars represent the reduced attribute counts; the right subfigure displays the attribute reduction rates for each dataset in percentage terms. Among these, the Handwritten Digits dataset demonstrates the most significant reduction effect: its original 64 attributes are reduced to 18, resulting in a net decrease of 46 attributes, which confirms the strong dimensionality reduction capability of the BIFIS algorithm for high-dimensional data. The Image Segmentation dataset shows a reduction of 10 attributes; although its absolute compression volume is lower than that of the Handwritten Digits dataset, it still exhibits high efficiency for moderately dimensional data. The attribute reduction rates in the right subfigure highly align with the pattern observed in the left subfigure that "the higher the original attribute count, the higher the reduction rate," indicating a significant positive correlation between the reduction rate of the BIFIS algorithm and the initial dimensionality of the dataset.

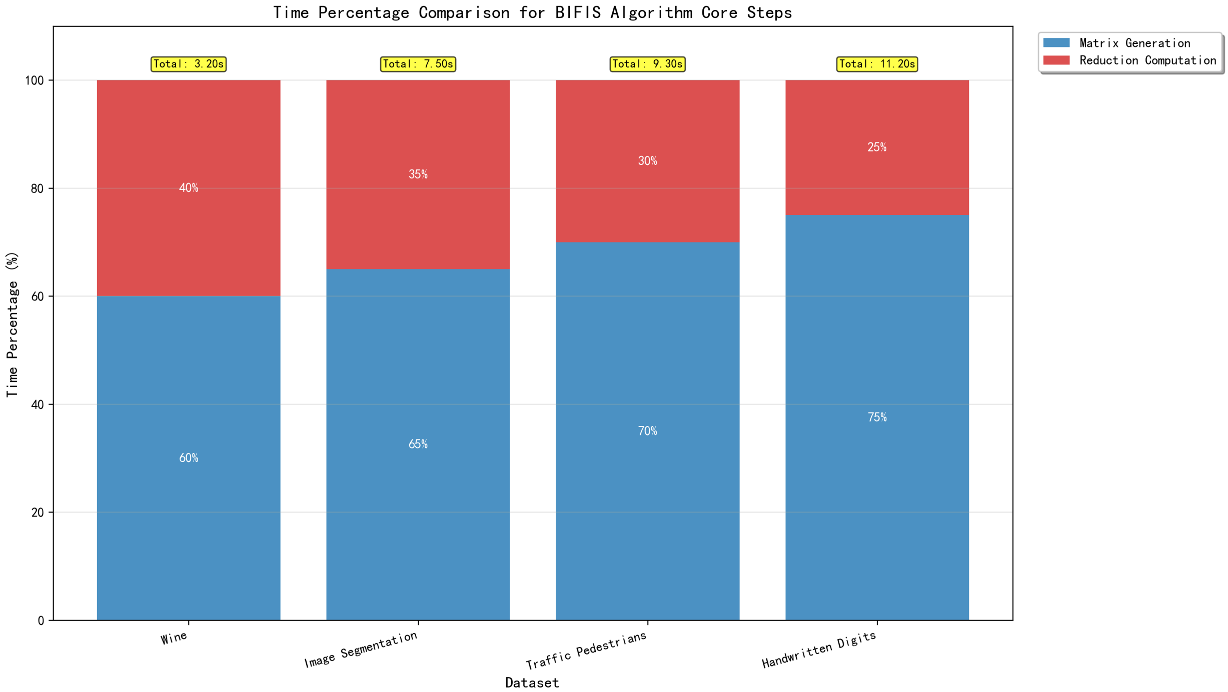

Fig. 1: Time proportion comparison for BIFIS algorithm core steps

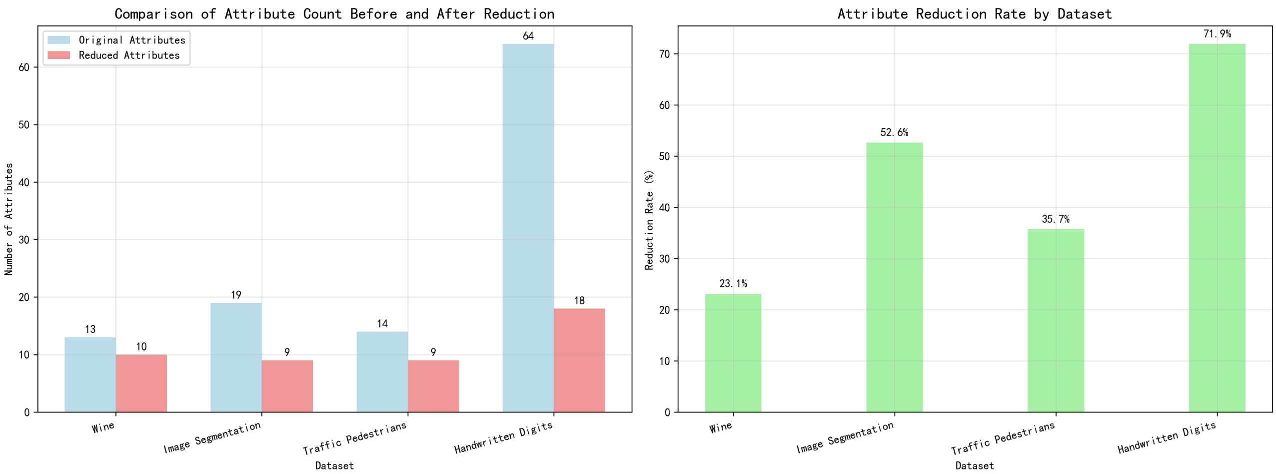

Fig. 2: Comparative effectiveness of attribute reduction

Conclusion

In the field of hybrid data processing, the attributes reductions of bipolar fuzzy relation decision systems proposed by Ali et al. (2020) represents a foundational contribution. Their work pioneered the processing of bipolar fuzzy data encompassing both positive and negative polarity information, with its core methodology relying on a single, global bipolar fuzzy relation to construct a discernibility matrix, which subsequently defines attribute reduction. Although groundbreaking in addressing data bipolarity, the theoretical framework exhibits limitations in characterizing complex characterizing complex uncertainties.

Ali's BFRDS model only accounts for positive and negative membership degrees of objects. In practice, however, decision makers often face incomplete information or hesitation-an attribute may be neither fully supported nor fully opposed. This paper presents a comprehensive framework for attribute reduction in bipolar intuitionistic fuzzy relation system (BIFRS), addressing the challenges of high-dimensional redundancy and spatial interactions in real-world data. By introducing positive and negative discernibility matrices (M^{+*} and M^{-*}), the proposed method systematically identifies minimal attribute subsets that preserve the bipolar consistency condition ($h^+ = O = h^-$). Theoretical proofs confirm that the global relation S_Q and the reduced relation S_O maintain complete consistency across all four components ($\pi^N, \pi^P, \eta^N, \eta^P$) if and only if O covers all discernibility terms.

The developed reduction algorithm optimizes matrix operations, reducing time complexity to polynomial time ($O(mn^2)$), making it efficient for practical applications. Experimental validation on real-world datasets demonstrates the algorithm's effectiveness and superiority in decision support and feature selection. Furthermore, the method extends reduction theories of traditional fuzzy and rough sets, providing interpretable tools for complex scenarios like medical diagnosis and investment risk analysis.

Future work could explore integrating spatial interactions and applying the framework to larger datasets, enhancing its scalability and robustness in diverse domains. This research not only advances the theoretical foundations of BIFRS but also offers practical solutions for managing uncertainty and bipolarity in information systems.

Ethics Statement

Conflict of Interest

The authors declare that they have no conflict of interest regarding the publication of this paper.

Authors Contributions

Weiyuan Wang: Conceptualization, Methodology, Software,validation, data curation, writing-original draft preparation.

Qian Wang: Conceptualization, supervision, project administration, funding acquisition, writing-review and editing. All authors have read and agreed to the published version of the manuscript.

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