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COMPLEX PROBABILITY THEORY AND PROGNOSTIC

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ABSTRACT

The Kolmogorov's system of axioms can be extended to encompass the imaginary set of numbers and this by adding to the original five axioms an additional three axioms. Hence, any experiment can thus be executed in what is now the complex set C (Real set R with real probability + Imaginary set M with imaginary probability). The objective here is to evaluate the complex probabilities by considering supplementary new imaginary dimensions to the event occurring in the "real" laboratory. Whatever the probability distribution of the input random variable in R is, the corresponding probability in the whole set C is always one, so the outcome of the random experiment in C can be predicted totally. The result indicates that chance and luck in R is replaced now by total determinism in C. This new complex probability model will be applied to the concepts of degradation and the Remaining Useful Lifetime (RUL), thus to the field of prognostic.

Keywords: Complex Probability, Probability Distributions, Prognostic, Degradation, Lifetime

1. INTRODUCTION

Abou Jaoude *et al*. (2010); Abou Jaoude (2013; 2005; 2007); Bell (1992); Benton (1996); Boursin (1986); Chen *et al*. (1997); Cheney and Kincaid (2004); Dacunha-Castelle (1999); Dalmédico Dahan *et al*. (1992); Dalmedico Dahan and Peiffer (1986); Ekeland (1991); Feller (1968); Finney *et al*. (2004); Gentle (2003); Gerald and Wheatley (1999); Gleick (1997); Greene (2000; 2004) firstly, the Extended Kolmogorov's Axioms (EKA for short) paradigm can be illustrated by the following figure (**Fig. 1**).

In engineering systems, the remaining useful lifetime prediction is related deeply to many factors that generally have a chaotic behavior which decreases the degree of our knowledge of the system.

As the Degree of Our Knowledge (DOK for short) in the real universe R is unfortunately incomplete, the extension to the complex universe C includes the contributions of both the real universe R and the imaginary universe M. Consequently, this will result in a complete and perfect degree of knowledge in C = R+M (Pc = 1). In fact, in order to have a certain prediction of any event it is necessary to work in the complex universe C in which the chaotic factor is quantified and subtracted from the Degree Of Knowledge to lead to a probability in C equal to one (Pc^2 = DOK-Chf = 1). Thus, the study in the complex universe results in replacing the phenomena that used to be random in R by deterministic and totally predictable ones in C.

This hypothesis is verified in a previous study and paper by the mean of many examples encompassing both discrete and continuous distributions.

From the Extended Kolmogorov's Axioms (EKA), we can deduce that if we add to an event probability in the real set R the imaginary part M (like the lifetime variables) then we can predict the exact probability of the remaining lifetime with certainty in C (Pc = 1).

We can apply this idea to prognostic analysis through the degradation evolution of a system. As a matter of fact, prognostic analysis consists in the prediction of the remaining useful lifetime of a system at any instant t_0 and during the system functioning.

Let us consider a degradation trajectory D(t) of a system where a specific instant t_0 is studied. The instant t_0 means here the time or age that can be measured also by the cycle number N.

Referring to the figure below (**Fig. 2**), the previous statement means that at the system age t_0, the prognostic study must give the prediction of the failure instant t_N. Therefore, the RUL predicted here at instant t_0 is the following quantity: RUL (t_0) = t_N-t_0.

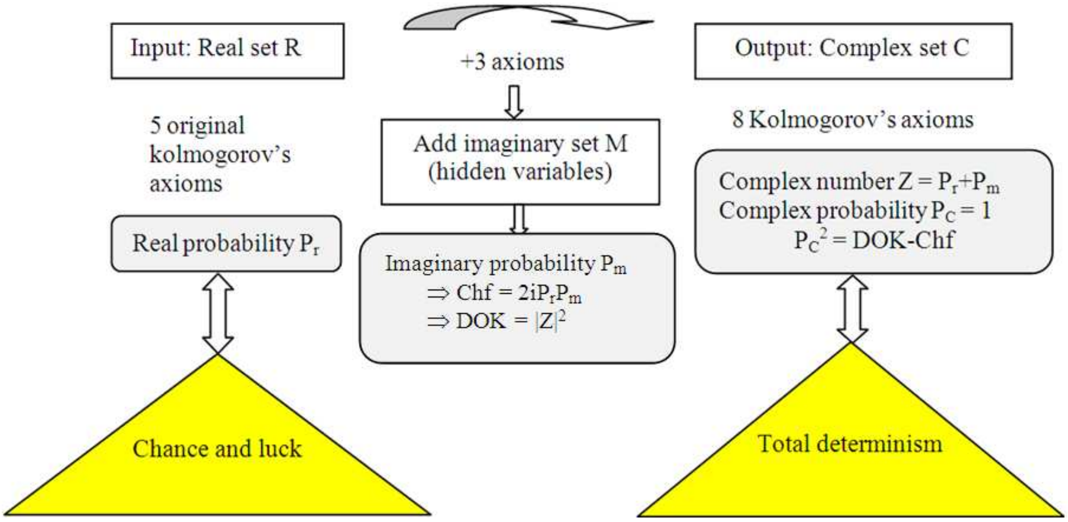

Fig. 1. EKA paradigm

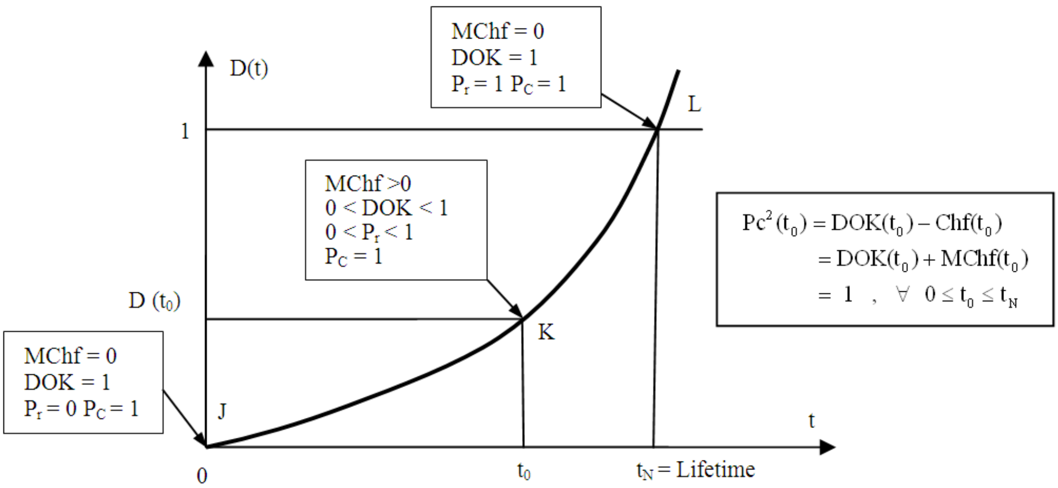

Fig. 2. EKA and the prognostic of degradation

In fact, at the beginning ($t_0 = 0$) (point J), the failure probability $P_r = 0$ and the chaotic factor in our prediction is zero (Chf = 0). Therefore, RUL($t_0 = 0$) = t_N-t_0 = t_N.

If $t_0 = t_N$ (point L) then the RUL(t_N) = t_N-t_N = 0 and the failure probability is one ($P_r = 1$).

If not (i.e., $0<t_0<t_N$) (point K), the probability of the occurrence of this instant and the prediction probability of RUL are both less than one (not certain) due to non-zero chaotic factors. The degree of our knowledge is consequently less than 1. Thus, by applying here the EKA method, we can determine the system RUL with certainty in C = R+M where Pc = 1 always.

Furthermore, we need in our current study the absolute value of the chaotic factor that will give us the magnitude of the chaotic and random effects on the studied system. This new term will be denoted accordingly MChf or Magnitude of the Chaotic Factor. Hence, we can deduce the following:

$$\begin{aligned}
&\text{MChf}(t_0) = |\text{Chf}(t_0)| \ge 0 \text{ and} \\
&Pc^2(t_0) = \text{DOK}(t_0) - \text{Chf}(t_0) = \text{DOK}(t_0) + |\text{Chf}(t_0)|, \\
&\text{since } -0.5 \le \text{Chf}(t_0) \le 0 \\
&= \text{DOK}(t_0) + \text{MChf}(t_0) = 1, \quad \forall \quad 0 \le t_0 \le t_N \\
&\Leftrightarrow 0 \le \text{MChf}(t_0) \le 0.5 \quad \text{where } 0.5 \le \text{DOK}(t_0) \le 1
\end{aligned}$$

Moreover, we can define two complementary events E and $\bar{E}$ with their respective probabilities:

$$P_{rob}(E) = p \quad \text{and} \quad P_{rob}(\bar{E}) = q = 1 - p$$

Then P_{rob} (E) in terms of the instant t_0 is given by: P_{rob} (E) = P_r = P_{rob} ($t \le t_0$) = $F(t_0)$ where F is the cumulative probability distribution function of the random variable t.

Since P_{rob} (E)+$P_{rob}(\bar{E})$= 1, therefore, $P_{rob}(\bar{E})$ = 1-$P_{rob}(E)$ = 1-P_r = 1-$P_{rob}(t \le t_0)$= P_{rob} ($t>t_0$).

Let us define the two particular instants: $t_0 = 0$ assumed as the initial time of functioning (raw state) corresponding to $D = D_0 = 0$ and t_N = the failure instant (wear out state) corresponding to the degradation D = 1.

The boundary conditions are.

For $t_0 = 0$ then $D = D_0$ (initial damage that may be zero or not) and F (t_0) = P_{rob} ($t \le 0$) = 0.

For $t_0 = t_N$ then D = 1 and $F(t_0)$ = F (t_N) = P_{rob} ($t \le t_N$) = 1.

Also $F(t_0)$ is a non-decreasing function that varies between 0 and 1. In fact, $F(t_0)$ is a cumulative function (**Fig. 3**). In addition, since $RUL(t_0) = t_N - t_0$ and $0 \le t_0 \le t_N$ then $RUL(t_0)$ is a non-increasing remaining useful lifetime function (**Fig. 4**).

Referring to **Fig. 5** below, we can infer the following:

The complex probability $Z(t_0) = P_r(t_0)+P_m(t_0) = P_r(t_0)+i[1-P_r(t_0)]$.

The square of the norm of $Z(t_0)$ is:

$$\begin{aligned}|Z(t_0)|^2 &= DOK(t_0) = 1 + 2iP_r(t_0)P_m(t_0) \\ &= 1 - 2P_r(t_0)\left[1 - P_r(t_0)\right] = 1 - 2P_r(t_0) + 2P_r^2(t_0)\end{aligned}$$

The Chaotic Factor and the Magnitude of the Chaotic Factor are:

$Chf(t_0) = -2P_r(t_0)[1-P_r(t_0)] = -2P_r(t_0)+2P_r^2(t_0)$ is null when $P_r(t_0) = P_r(0) = 0$ (point J) or when $P_r(t_0) = P_r(t_N) = 1$ (point L) and $MChf(t_0) = |Chf(t_0)| = 2P_r(t_0)[1-P_r(t_0)] = 2P_r(t_0)-2P_r^2(t_0)$ is null when $P_r(t_0) = P_r(0) = 0$ (point J) or when $P_r(t_0) = P_r(t_N) = 1$ (point L)

At any instant t_0 (point K), the probability expressed in the complex set C is:

$Pc(t_0) = P_r(t_0)+P_m(t_0)/i = P_r(t_0)+[1-P_r(t_0)] = 1$ always.

Hence, the prediction of $RUL(t_0)$ of the system degradation in C is permanently certain.

Let us consider thereafter many probability distributions to model the function $F(t_0)$.

2. APPLICATION TO DIFFERENT PROBABILITY DISTRIBUTIONS

2.1. The Uniform Probability Distribution (Guillen, 1995; Gullberg, 1997; Kuhn, 1996; Liu, 2001; Mandelbrot, 1997; Montgomery and Runger, 2005; Mũller, 2005; Orluc and Poirier, 2005; Poincaré, 1968; Prigogine, 1997; Prigogine and Stengers, 1992; Robert and Casella, 2010; Science et Vie, 1999; Srinivasan and Mehata, 1978; Stewart, 1996; 2002; Van Kampen, 2007; Walpole, 2002; Ducrocq and Warusfel, 2004; Weinberg, 1992)

With a probability density function:

$$f(t) = \frac{dF(t)}{dt} = \begin{cases} \dfrac{1}{b-a} & \text{if} \quad a \le t \le b \\ 0 & \text{elsewhere} \end{cases}$$

and a cumulative distribution function:

$$F(t_0) = P_{rob}(t \le t_0) = \int_{-\infty}^{t_0} f(t)dt = \int_{a}^{t_0} f(t)dt = \begin{cases} \dfrac{t_0 - a}{b-a} & \text{if} \quad a \le t_0 \le b \\ 0 & \text{elsewhere} \end{cases}$$

With the two boundaries a = 0 and b = t_N then:

$$F(t_0) = \frac{t_0 - 0}{t_N - 0} = \frac{t_0}{t_N} \quad \text{if} \quad 0 \le t_0 \le t_N$$

We have taken the domain for the uniform variable t_0= [0, t_N = 1000] and dt_0 = 0.1 then:

$$F(t_0) = \frac{t_0}{1000} \quad \text{if} \quad 0 \le t_0 \le 1000$$

2.1.1. The Real Probability P_r:

$$P_r(t_0) = F(t_0) = \frac{t_0}{t_N} \quad \text{if} \quad 0 \le t_0 \le t_N = 1000$$

We note that $P_r(t_0)$ is a non-decreasing function.

2.1.2. The Complementary Probability P_m/i:

$$P_m(t_0)/i = 1 - P_r(t_0) = 1 - F(t_0) = 1 - \frac{t_0}{t_N} \quad \text{if} \quad 0 \le t_0 \le t_N = 1000$$

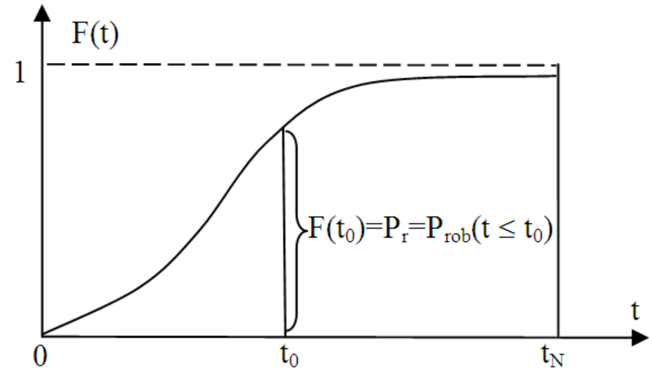

Fig. 3. Occurrence probability

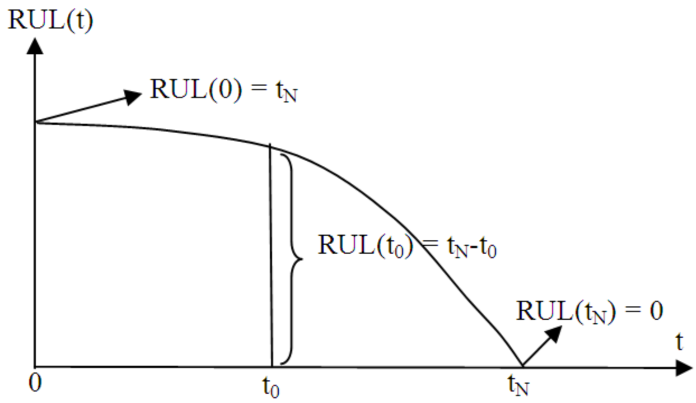

Fig. 4. RUL prognostic model

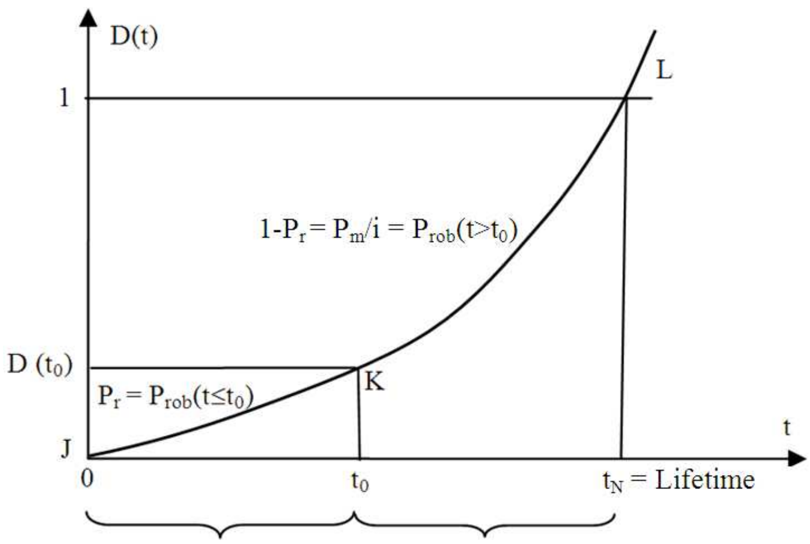

Fig. 5. Degradation prognostic model

We note that $P_m(t_0)/i$ is a non-increasing function.

2.1.3. The Degree of Our Knowledge DOK

DOK is the measure of our certain knowledge (100% probability) about the expected event, it does not include any uncertain knowledge (with probability less than 100%):

$$\begin{aligned}
DOK(t_0) &= Pc^2(t_0) + 2iP_r(t_0)P_m(t_0) \\
&= 1 - 2.P_{rob}(E).P_{rob}(\bar{E}) = 1 - 2.P_r(t_0).[1 - P_r(t_0)] \\
&= 1 - 2.F(t_0).[1 - F(t_0)] = 1 - 2.\frac{t_0}{t_N}.\left(1 - \frac{t_0}{t_N}\right) \\
&= 1 - 2.\frac{t_0}{t_N} + 2.\left(\frac{t_0}{t_N}\right)^2
\end{aligned}$$

Which is a parabola concave upward having a vertex (a minimum) at:

$$\left(t_0 = \frac{t_N}{2} = 500 = 0.5 \times 10^3,\ \ 0.5\right)$$

2.1.4. The Chaotic Factor Chf and MChf:

$$\begin{aligned}
Chf(t_0) &= 2iP_r(t_0)P_m(t_0) = -2.P_{rob}(E).P_{rob}(\bar{E}) \\
&= -2.P_r(t_0).[1 - P_r(t_0)] = -2.F(t_0).[1 - F(t_0)] \\
&= -2.\frac{t_0}{t_N}.\left(1 - \frac{t_0}{t_N}\right) = -2.\frac{t_0}{t_N} + 2.\left(\frac{t_0}{t_N}\right)^2
\end{aligned}$$

Which is a parabola concave upward having a vertex (a minimum) at:

$$\left(t_0 = \frac{t_N}{2} = 500 = 0.5 \times 10^3,\ \ -0.5\right)$$

Therefore, we can infer the magnitude of the chaotic factor MChf:

$$\begin{aligned}
MChf(t_0) &= |Chf(t_0)| = |-2.P_r(t_0).[1 - P_r(t_0)]| \\
&= 2.P_r(t_0).[1 - P_r(t_0)] = 2.F(t_0).[1 - F(t_0)] \\
&= 2.\frac{t_0}{t_N}.\left(1 - \frac{t_0}{t_N}\right) = 2.\frac{t_0}{t_N} - 2.\left(\frac{t_0}{t_N}\right)^2
\end{aligned}$$

Which is a parabola concave downward having a vertex (a maximum) at:

$$\left(t_0 = \frac{t_N}{2} = 500 = 0.5 \times 10^3,\ \ 0.5\right)$$

2.1.5. Pc: The Probability in the Complex Set C:

$$\begin{aligned}
Pc^2(t_0) &= DOK(t_0) - Chf(t_0) \\
&= 1 - 2.\frac{t_0}{t_N} + 2.\left(\frac{t_0}{t_N}\right)^2 + 2.\frac{t_0}{t_N} - 2.\left(\frac{t_0}{t_N}\right)^2 = 1 \Rightarrow Pc(t_0) = 1
\end{aligned}$$

Thus we deduce that in the set C, we have a complete knowledge of the random variable since Pc = 1.

2.1.6. The Intersection Point:

$$\begin{aligned}
&P_r(t_0) = P_m(t_0)/i \Leftrightarrow \frac{t_0}{t_N} = 1 - \frac{t_0}{t_N} \Leftrightarrow 2.\frac{t_0}{t_N} \\
&= 1 \Leftrightarrow t_0 = \frac{t_N}{2} = \frac{1000}{2} = 500 \\
&\text{and } P_r(t_0 = 500) = \frac{500}{1000} = 0.5 \ \ \text{and} \\
&P_m(t_0 = 500)/i = 1 - \frac{500}{1000} = 1 - 0.5 = 0.5
\end{aligned}$$

So $P_r(t_0)$ and $P_m(t_0)/i$ intersect at (500, 0.5).

Moreover, the minimum of DOK and the maximum of MChf occur at (500, 0.5).

So we conclude that $P_r(t_0)$, $P_m(t_0)/i$, DOK and MChf all intersect at (500, 0.5) (**Fig. 6-9**).

2.1.7. The EKA Parameters Analysis in the Prognostic of Degradation:

We note from the figure below that the DOK is maximum (DOK = 1) when MChf is minimum (MChf = 0) (points J & L) and that means when the magnitude of the chaotic factor (MChf) decreases our certain knowledge increases.

At the beginning $P_r(t_0) = t_0/t_N = 0/t_N = 0$, the system is intact (zero damage: D = 0) and has zero chaotic factor before any usage, at this instant DOK(0) = 1 and RUL(0) = t_N-0 = t_N with Pc(0) = 1. Afterward, $0<t_0<t_N$, $RUL(t_0) = t_N - t_0$ with $P_r(t_0) = t_0/t_N \neq 0$ and $Pc(t_0) = 1$ and MChf starts to increase during the functioning due to the environment and intrinsic conditions thus leading to a decrease in DOK until they both reach 0.5 at $t_0 = t_N/2 = 500$ (point K) where $RUL(t_N/2) = t_N/2 = 500$. Since the real probability P_r is a uniform distribution MChf will intersect with DOK at the point ($t_N/2 = 500$, 0.5) (point K).

With the increase of the time of functioning, MChf returns to zero and the DOK returns to 1 where we reach total damage (D = 1) and hence the total failure of the system (point L). At this last point the failure here is certain, $P_r(t_N) = t_N/t_N = 1$ and $RUL(t_N) = t_N - t_N = 0$ with $Pc(t_N) = 1$, so the logical explanation of the value DOK = 1 follows (**Fig. 10**).

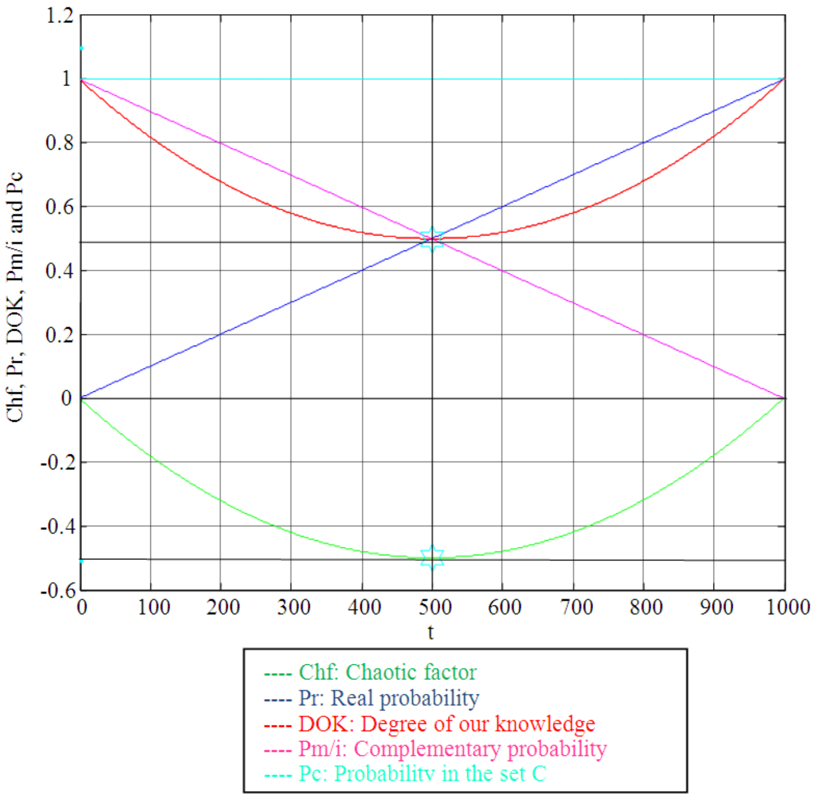

Fig. 6. EKA parameters in uniform probability distribution

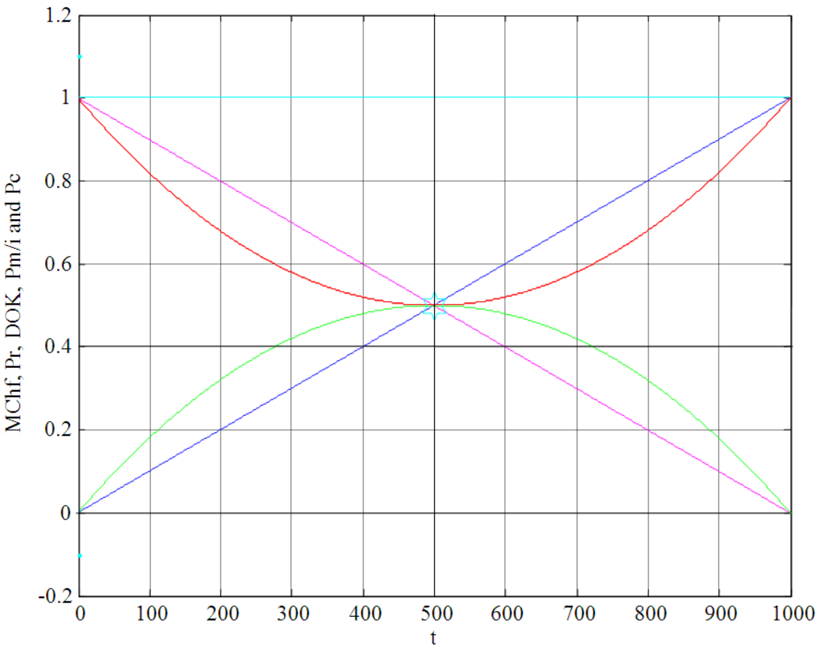

Fig. 7. EKA parameters in uniform probability distribution

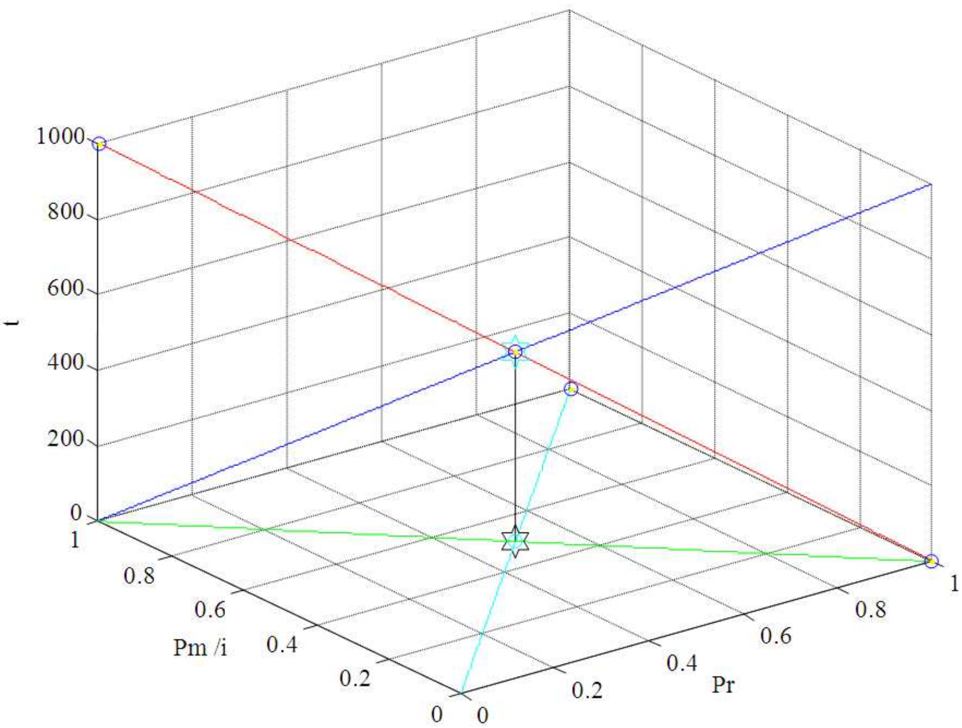

Fig. 8. The probabilities Pr and Pm / i in terms of t and of each other

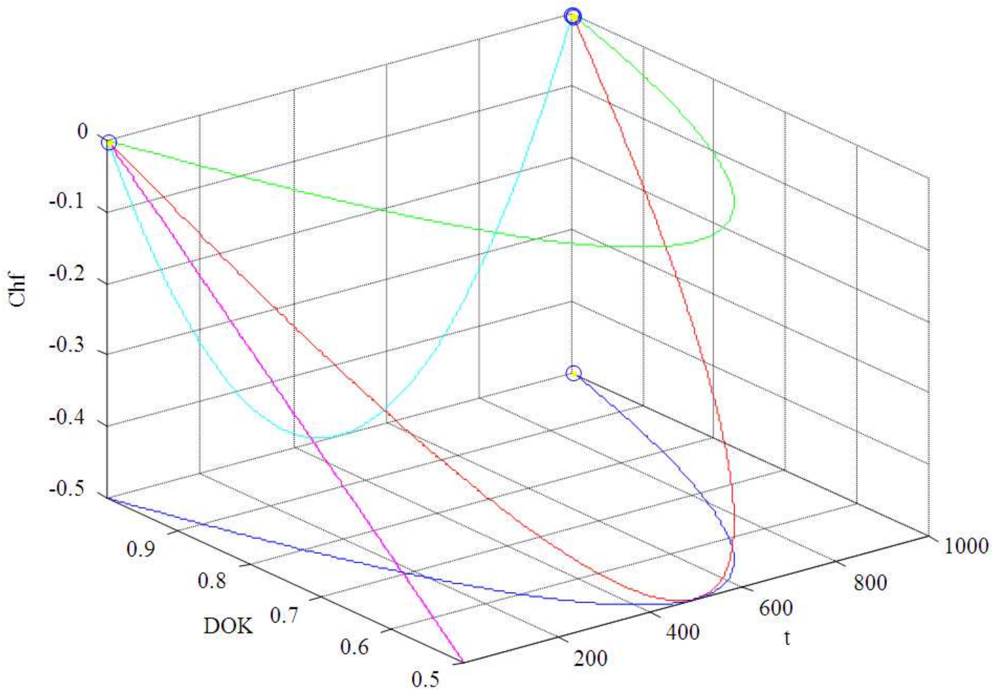

Fig. 9. DOK and Chf in terms of t and of each other in uniform probability distribution

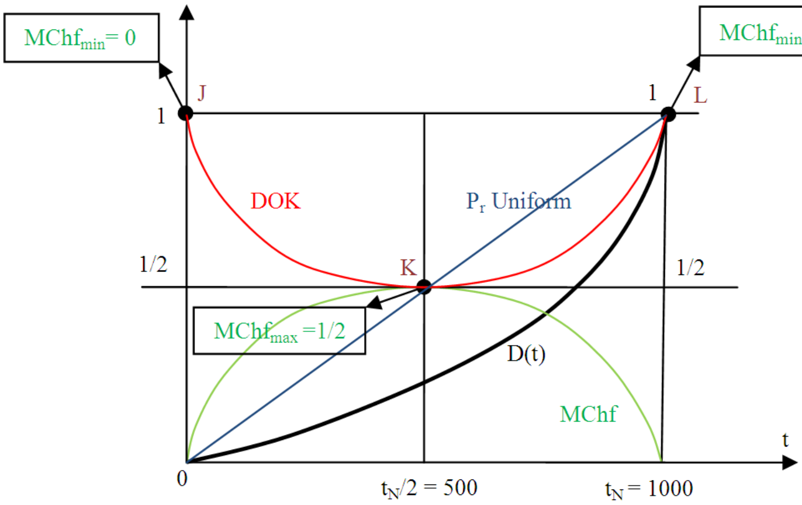

Fig. 10. EKA parameters and the prognostic of degradation

We note that the same logic and analysis apply concerning the degradation and the remaining useful lifetime for the all the six probability distributions.

2.2. The Logarithmic Distribution:

With a probability density function:

$$f(t) = \frac{dF(t)}{dt} = \begin{cases} \dfrac{\alpha}{1+\alpha t} & \text{if} \quad 0 \le t \le t_N \\ 0 & \text{elsewhere} \end{cases}$$

$$\text{where } \alpha = (e-1)/1000 = 0.001718281...$$

and a cumulative distribution function:

$$F(t_0) = \int_{-\infty}^{t_0} f(t)dt = \int_0^{t_0} f(t)dt = \begin{cases} Ln(1+\alpha t_0) & \text{if} \quad 0 \le t_0 \le t_N \\ 0 & \text{elsewhere} \end{cases}$$

We have taken the domain for the logarithmic variable t_0= [0, t_N = 1000] where dt_0 = 0.1.

2.2.1. The Real Probability P_r:

$$P_r = F(t_0) = Ln(1+\alpha t_0) \quad \text{if} \quad 0 \le t_0 \le t_N = 1000$$

We note that P_r is a non-decreasing function.

2.2.2. The Complementary Probability P_m/i:

$$P_m / i = 1 - P_r = 1 - F(t_0) = 1 - Ln(1+\alpha t_0)$$
$$\text{if} \quad 0 \le t_0 \le t_N = 1000$$

We note that P_m/i is a non-increasing function.

2.2.3. The Degree of Our Knowledge DOK:

DOK is the measure of our certain knowledge (100% probability) about the expected event, it does not include any uncertain knowledge (with probability less than 100%):

$$\begin{aligned} DOK &= 1 - 2.P_r.(1-P_r) = 1 - 2.P_{rob}(E).P_{rob}(\bar{E}) \\ &= 1 - 2.F(t_0).[1 - F(t_0)] \\ &= 1 - 2.Ln(1+\alpha t_0).[1 - Ln(1+\alpha t_0)] \\ &= 1 - 2.Ln(1+\alpha t_0) + 2[Ln(1+\alpha t_0)]^2 \end{aligned}$$

which is a curve concave upward having a minimum at:

$$(t_0 = 377.54, \quad 0.5)$$

2.2.4. The Chaotic Factor Chf and MChf:

$$\begin{aligned} Chf &= -2.P_r.(1-P_r) = -2.P_{rob}(E).P_{rob}(\bar{E}) \\ &= -2.F(t_0).[1 - F(t_0)] \\ &= -2.Ln(1+\alpha t_0).[1 - Ln(1+\alpha t_0)] \\ &= -2.Ln(1+\alpha t_0) + 2.[Ln(1+\alpha t_0)]^2 \end{aligned}$$

which is a curve concave upward having a minimum at:

$$(t_0 = 377.54, \quad -0.5)$$

Therefore, we can infer the magnitude of the chaotic factor MChf:

$$\begin{aligned}\mathrm{MChf}(t_0) &= |\mathrm{Chf}(t_0)| = |-2.P_r(t_0).[1-P_r(t_0)]| \\ &= 2.P_r(t_0).[1-P_r(t_0)] = 2.F(t_0).[1-F(t_0)] \\ &= 2.\mathrm{Ln}(1+\alpha t_0).[1-\mathrm{Ln}(1+\alpha t_0)] \\ &= 2.\mathrm{Ln}(1+\alpha t_0) - 2.[\mathrm{Ln}(1+\alpha t_0)]^2\end{aligned}$$

Which is a curve concave downward having a maximum at:

$$(t_0 = 377.54,\ \ 0.5)$$

2.2.5. Pc: The Probability in the Complex Set C:

$$\begin{aligned}&Pc^2 = \mathrm{DOK} - \mathrm{Chf} = 1 - 2.\ln(1+\alpha t_0) + 2.[\ln(1+\alpha t_0)]^2 \\ &+2.\ln(1+\alpha t_0) - 2.[\ln(1+\alpha t_0)]^2 = 1 \Rightarrow Pc = 1\end{aligned}$$

Thus we deduce that in the set C, we have a complete knowledge of the random variable since Pc = 1.

2.2.6. The Intersection Point:

$$\begin{aligned}&P_r(t_0) = P_m(t_0)/i \Leftrightarrow \mathrm{Ln}(1+\alpha t_0) = 1 - \mathrm{Ln}(1+\alpha t_0) \\ &\Leftrightarrow 2.\mathrm{Ln}(1+\alpha t_0) = 1 \Leftrightarrow \mathrm{Ln}(1+\alpha t_0) = \frac{1}{2} \\ &\Leftrightarrow t_0 = (\exp(0.5)-1)/\alpha = 377.54 \\ &\text{and } P_r(t_0 = 377.54) = \mathrm{Ln}(1+\alpha\times 377.54) = 0.5 \\ &\text{and } P_m(t_0 = 377.54)/i = 1 - \mathrm{Ln}(1+\alpha\times 377.54) \\ &= 1 - 0.5 = 0.5\end{aligned}$$

So $P_r(t_0)$ and $P_m(t_0)/i$ intersect at (377.54, 0.5).

Moreover, the minimum of DOK and the maximum of MChf occur at (377.54, 0.5).

So we conclude that $P_r(t_0)$, $P_m(t_0)/i$, DOK and MChf all intersect at (377.54, 0.5) (**Fig. 11-13**).

2.2.7. The EKA Parameters Analysis in the Prognostic of Degradation:

In this case, we note from the figure below that the DOK is maximum (DOK = 1) when MChf is minimum (MChf = 0) (points J & L). Afterward, MChf starts to increase with the decrease of DOK until it reaches 0.5 at t_0 = 377.54 (point K). Since the real probability P_r is a logarithmic distribution (convex curve) it will intersect with DOK at the point (377.54, 0.5) (point K). With the increase of t_0, MChf returns to zero and the DOK returns to 1 where we reach total damage (D = 1) and hence the total certain failure (P_r = 1) of the system (point L). We note that the point K is no more at the middle of DOK since the distribution is not anymore uniform and symmetric.

At each instant t_0, the remaining useful lifetime RUL(t_0) is certainly predicted in the complex set C with Pc maintained as equal to one through a continuous compensation between DOK and Chf. This compensation is from instant $t_0 = 0$ where $D(t_0) = 0$ until the failure instant t_N where $D(t_N) = 1$ (**Fig. 11**).

2.3. The Power Probability Distribution:

With a probability density function:

$$f(t) = \frac{dF(t)}{dt} = \begin{cases} \dfrac{14.t^{13}}{1000^{14}} & \text{if} \quad 0 \le t \le t_N \\ 0 & \text{elsewhere}\end{cases}$$

and a cumulative distribution function:

$$F(t_0) = \int_{-\infty}^{t_0} f(t)dt = \int_0^{t_0} f(t)dt = \begin{cases} \dfrac{t_0^{14}}{1000^{14}} & \text{if} \quad 0 \le t_0 \le t_N \\ 0 & \text{elsewhere}\end{cases}$$

We have taken the domain for the power variable $t_0 = [0, t_N = 1000]$ and $dt_0 = 0.1$.

2.3.1. The Real Probability P_r:

$$P_r = F(t_0) = \frac{t_0^{14}}{1000^{14}} \quad \text{if} \quad 0 \le t_0 \le t_N = 1000$$

We note that P_r is a non-decreasing function.

2.3.2. The Complementary Probability P_m/i:

$$\begin{aligned}&P_m/i = 1 - P_r = 1 - F(t_0) = 1 - \frac{t_0^{14}}{1000^{14}} \\ &\text{if} \quad 0 \le t_0 \le t_N = 1000\end{aligned}$$

We note that P_m/i is a non-increasing function.

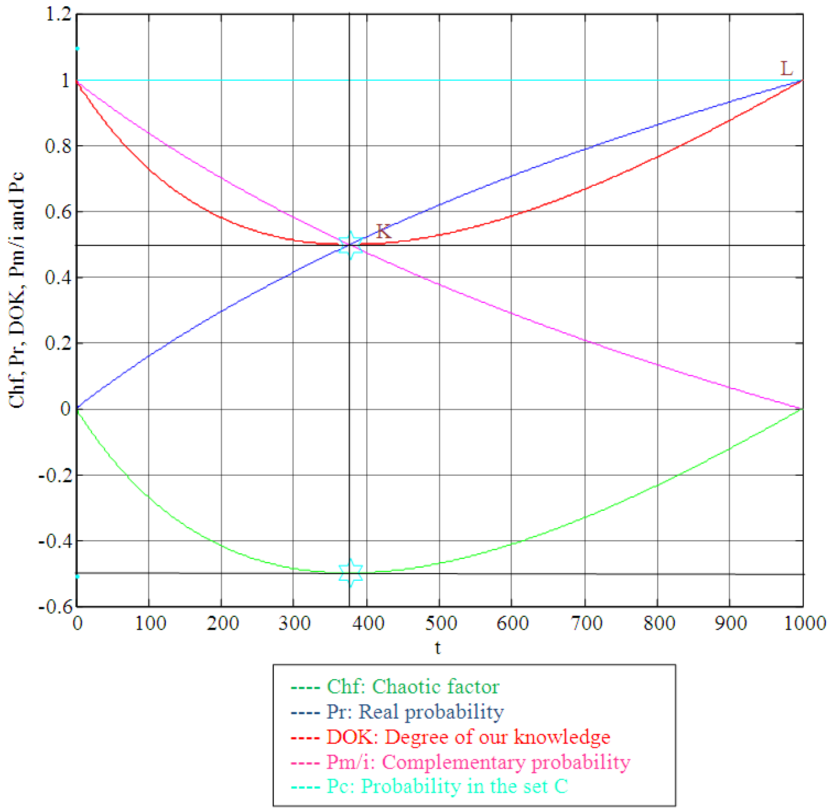

Fig. 11. EKA parameters in logarithmic cumulative distribution

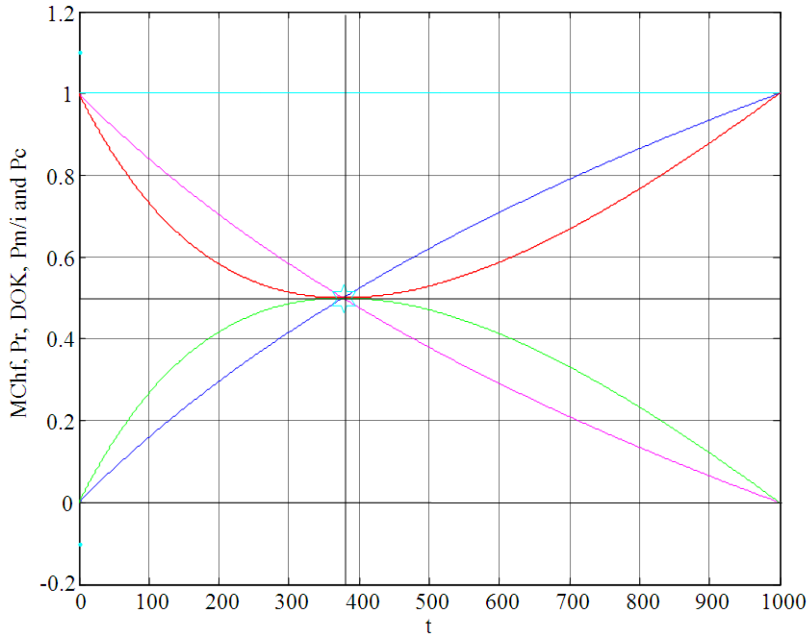

Fig. 12. EKA parameters in logarithmic cumulative distribution

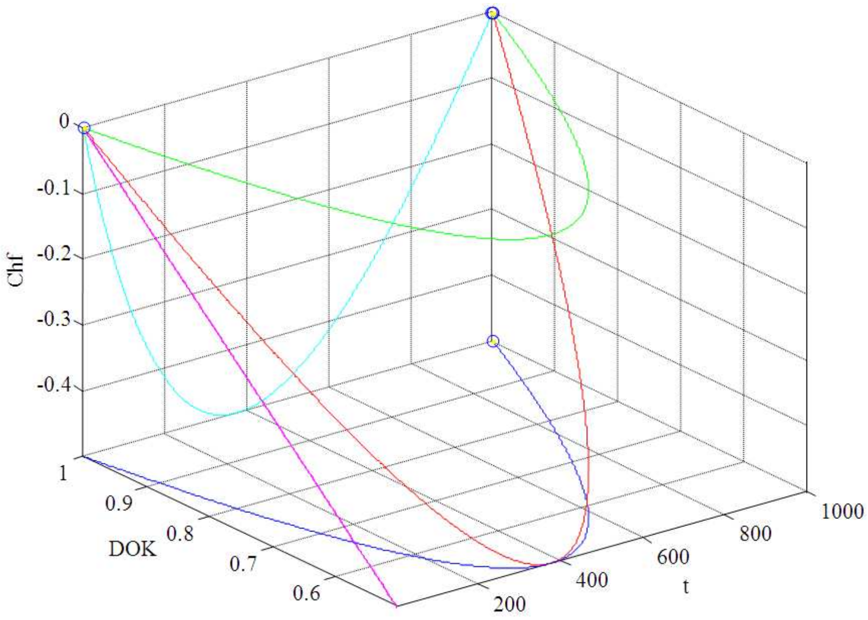

Fig. 13. DOK and Chf in terms of t and of each other in logarithmic cumulative distribution

2.3.3. The Degree of Our Knowledge DOK:

DOK is the measure of our certain knowledge (100% probability) about the expected event, it does not include any uncertain knowledge (with probability less than 100 %):

$$\begin{aligned}DOK &= 1-2.P_r.(1-P_r) = 1-2.P_{rob}(E).P_{rob}(\bar{E})\\ &= 1-2.F(t_0).[1-F(t_0)]\\ &= 1-2.\frac{t_0^{14}}{1000^{14}}.\left(1-\frac{t_0^{14}}{1000^{14}}\right) = 1-2.\frac{t_0^{14}}{1000^{14}}\\ &+2.\left(\frac{t_0^{14}}{1000^{14}}\right)^2 = 1-2.\frac{t_0^{14}}{1000^{14}}+2.\frac{t_0^{28}}{1000^{28}}\end{aligned}$$

Which is a curve concave upward having a minimum at:

$$\left(t_0 = 951.7,\ \ 0.5\right)$$

2.3.4. The Chaotic Factor Chf and MChf:

$$\begin{aligned}Chf &= -2.P_r.(1-P_r) = -2.P_{rob}(E).P_{rob}(\bar{E})\\ &= -2.F(t_0).[1-F(t_0)]\\ &= -2.\frac{t_0^{14}}{1000^{14}}.\left(1-\frac{t_0^{14}}{1000^{14}}\right) = -2.\frac{t_0^{14}}{1000^{14}}+2.\left(\frac{t_0^{14}}{1000^{14}}\right)^2\\ &= -2.\frac{t_0^{14}}{1000^{14}}+2.\frac{t_0^{28}}{1000^{28}}\end{aligned}$$

Which is a curve concave upward having a minimum at:

$$\left(t_0 = 951.7,\ \ -0.5\right)$$

Therefore, we can infer the magnitude of the chaotic factor MChf:

$$\begin{aligned}MChf(t_0) &= |Chf(t_0)| = |-2.P_r(t_0).[1-P_r(t_0)]|\\ &= 2.P_r(t_0).[1-P_r(t_0)] = 2.F(t_0).[1-F(t_0)]\\ &= 2.\frac{t_0^{14}}{1000^{14}}.\left(1-\frac{t_0^{14}}{1000^{14}}\right) = 2.\frac{t_0^{14}}{1000^{14}}-2.\frac{t_0^{28}}{1000^{28}}\end{aligned}$$

Which is a curve concave downward having a maximum at:

$$\left(t_0 = 951.7,\ \ 0.5\right)$$

2.3.5. Pc: The Probability in the Complex Set C:

$$\begin{aligned}Pc^2 &= DOK - Chf = 1-2.\frac{t_0^{14}}{1000^{14}}+2.\left(\frac{t_0^{14}}{1000^{14}}\right)^2\\ &+2.\frac{t_0^{14}}{1000^{14}}-2.\left(\frac{t_0^{14}}{1000^{14}}\right)^2 = 1 \Rightarrow Pc = 1\end{aligned}$$

Thus we deduce that in the set C, we have a complete knowledge of the random variable since Pc = 1.

2.3.6. The Intersection Point:

$$
\begin{aligned}
&P_r(t_0) = P_m(t_0)/i \Leftrightarrow \frac{t_0^{14}}{1000^{14}} = 1 - \frac{t_0^{14}}{1000^{14}} \\
&\Leftrightarrow 2.\frac{t_0^{14}}{1000^{14}} = 1 \Leftrightarrow \frac{t_0^{14}}{1000^{14}} = \frac{1}{2} \\
&\Leftrightarrow t_0 = \sqrt[14]{0.5 \times 1000^{14}} = 951.7 \\
&\text{and } P_r(t_0 = 951.7) = \frac{951.7^{14}}{1000^{14}} = 0.5 \\
&\text{and } P_m(t_0 = 951.7)/i = 1 - \frac{951.7^{14}}{1000^{14}} = 1 - 0.5 = 0.5
\end{aligned}
$$

So $P_r(t_0)$ and $P_m(t_0)/i$ intersect at (951.7, 0.5).

Moreover, the minimum of DOK and the maximum of MChf occur at (951.7, 0.5).

So we conclude that $P_r(t_0)$, $P_m(t_0)/i$, DOK and MChf all intersect at (951.7, 0.5) (**Fig. 14-16**).

2.3.7. The EKA Parameters Analysis in the Prognostic of Degradation:

In this case, we note from the figure below that the DOK is maximum (DOK=1) when MChf is minimum (MChf = 0) (points J & L). Afterward, MChf starts to increase with the decrease of DOK until it reaches 0.5 at $t_0 = 951.7$ (point K). Since the real probability P_r is a power distribution it will intersect with DOK at the point (951.7, 0.5) (point K). With the increase of t_0, MChf returns to zero and the DOK returns to 1 where we reach total damage (D =1) and hence the total certain failure ($P_r = 1$) of the system (point L). At this last point L the failure here is certain, $P_r(t_N) = t_N/t_N = 1$ and $RUL(t_N) = t_N - t_N = 0$ with $Pc(t_N) = 1$, so the logical explanation of the value DOK = 1 follows. We note that the point K is no more at the middle of DOK since the distribution is neither uniform nor symmetric (**Fig. 14**).

2.4. The Exponential Probability Distribution:

With a probability density function:

$$
f(t) = \frac{dF(t)}{dt} = \begin{cases} \frac{1}{200}\exp\left[-\frac{(1000-t)}{200}\right] & \text{if} \quad 0 \le t \le t_N \\ 0 & \text{elsewhere} \end{cases}
$$

and a cumulative distribution function:

$$
F(t_0) = \int_{-\infty}^{t_0} f(t)dt = \int_{0}^{t_0} f(t)dt
$$

$$
= \begin{cases} \exp\left[-\frac{(1000-t_0)}{200}\right] & \text{if} \quad 0 \le t_0 \le t_N \\ 0 & \text{elsewhere} \end{cases}
$$

We have taken the domain for the logarithmic variable $t_0 = [0, t_N = 1000]$ and $dt_0 = 0.1$.

2.4.1. The Real Probability P_r:

$$
P_r = F(t_0) = \exp\left[-\frac{(1000-t_0)}{200}\right] \quad \text{if} \quad 0 \le t_0 \le t_N = 1000
$$

We note that P_r is a non-decreasing function.

2.4.2. The Complementary Probability P_m/i:

$$
P_m/i = 1 - P_r = 1 - F(t_0) = 1 - \exp\left[-\frac{(1000-t_0)}{200}\right] \quad \text{if} \quad 0 \le t_0 \le t_N = 1000
$$

We note that P_m/i is a non-increasing function.

2.4.3. The Degree of Our Knowledge DOK:

DOK is the measure of our certain knowledge (100% probability) about the expected event, it does not include any uncertain knowledge (with probability less than 100%):

$$
\begin{aligned}
DOK &= 1 - 2.P_r.(1-P_r) = 1 - 2.P_{rob}(E).P_{rob}(\bar{E}) \\
&= 1 - 2.F(t_0).[1 - F(t_0)] \\
&= 1 - 2.\exp\left[-\frac{(1000-t_0)}{200}\right].\left(1 - \exp\left[-\frac{(1000-t_0)}{200}\right]\right) \\
&= 1 - 2.\exp\left[-\frac{(1000-t_0)}{200}\right] + 2.\left(\exp\left[-\frac{(1000-t_0)}{200}\right]\right)^2 \\
&= 1 - 2.\exp\left[-\frac{(1000-t_0)}{200}\right] + 2.\exp\left[-\frac{2.(1000-t_0)}{200}\right]
\end{aligned}
$$

Which is a curve concave upward having a minimum at:

$$
(t_0 = 861.37, \ 0.5)
$$

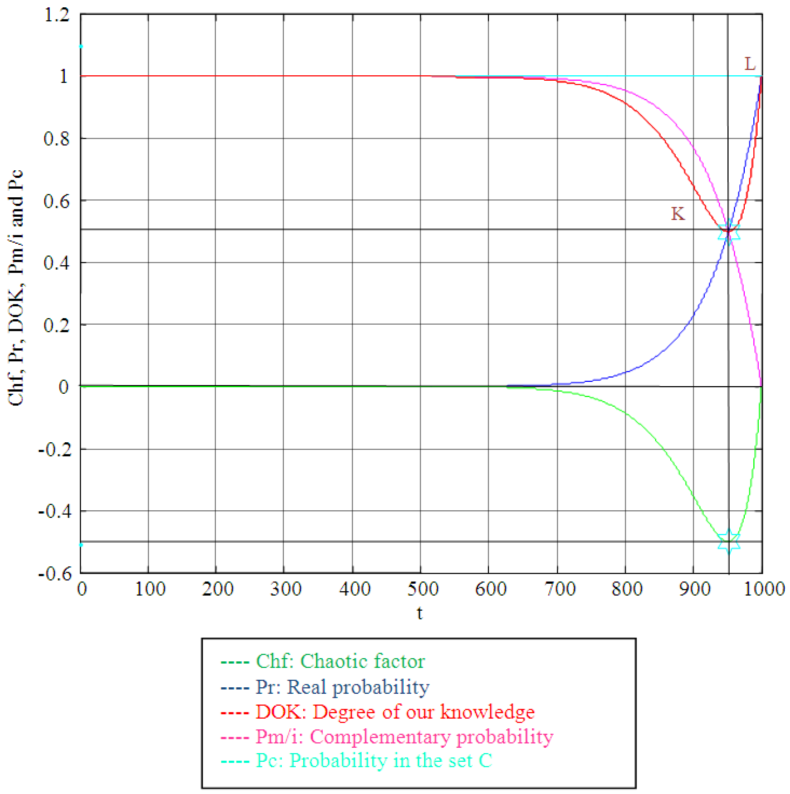

Fig. 14. EKA parameters in power probability distribution

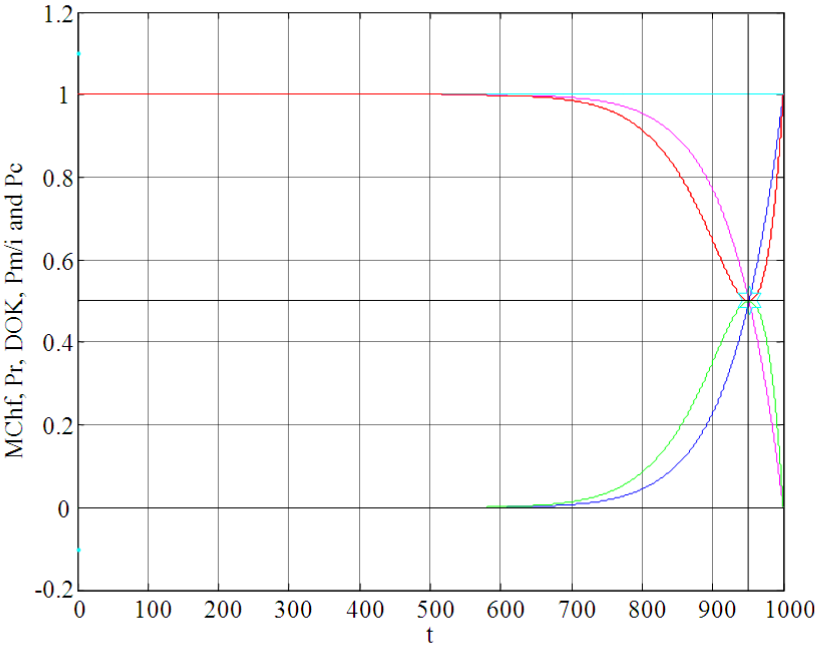

Fig. 15. EKA parameters in power cumulative distribution

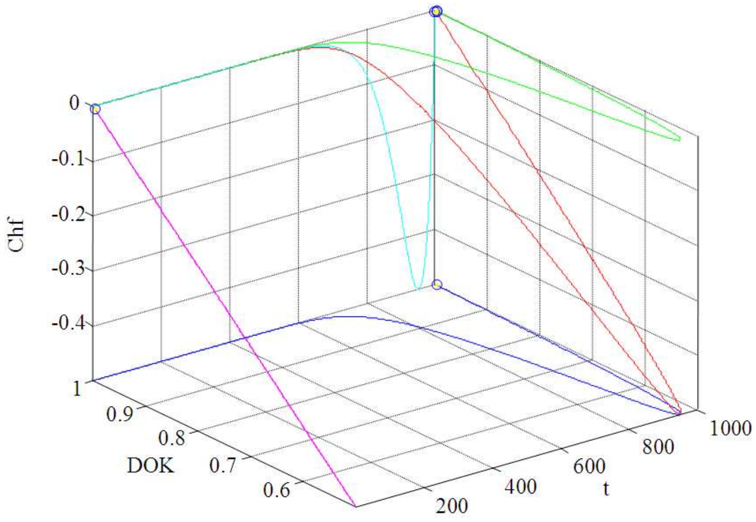

Fig. 16. DOK and Chf in terms of t and of each other in power probability distribution

2.4.4. The Chaotic Factor Chf and MChf:

$$\begin{aligned}
\text{Chf} &= -2.P_r.(1-P_r) = -2.P_{rob}(E).P_{rob}(\bar{E}) \\
&= -2.F(t_0).[1-F(t_0)] \\
&= -2.\exp\left[-\frac{(1000-t_0)}{200}\right].\left(1-\exp\left[-\frac{(1000-t_0)}{200}\right]\right) \\
&= -2.\exp\left[-\frac{(1000-t_0)}{200}\right] + 2.\left(\exp\left[-\frac{(1000-t_0)}{200}\right]\right)^2 \\
&= -2.\exp\left[-\frac{(1000-t_0)}{200}\right] + 2.\exp\left[-\frac{2.(1000-t_0)}{200}\right]
\end{aligned}$$

Which is a curve concave upward having a minimum at:

$$(t_0 = 861.37, \quad -0.5)$$

Therefore, we can infer the magnitude of the chaotic factor MChf:

$$\begin{aligned}
\text{MChf}(t_0) &= |\text{Chf}(t_0)| = |-2.P_r(t_0).[1-P_r(t_0)]| \\
&= 2.P_r(t_0).[1-P_r(t_0)] = 2.F(t_0).[1-F(t_0)] \\
&= 2.\exp\left[-\frac{(1000-t_0)}{200}\right].\left(1-\exp\left[-\frac{(1000-t_0)}{200}\right]\right) \\
&= 2.\exp\left[-\frac{(1000-t_0)}{200}\right] - 2.\exp\left[-\frac{2.(1000-t_0)}{200}\right]
\end{aligned}$$

Which is a curve concave downward having a maximum at:

$$(t_0 = 861.37, \quad 0.5)$$

2.4.5. Pc: The Probability in the Complex Set C:

$$\begin{aligned}
Pc^2 &= \text{DOK} - \text{Chf} = 1 - 2.\exp\left[-\frac{(1000-t_0)}{200}\right] \\
&+2.\left(\exp\left[-\frac{(1000-t_0)}{200}\right]\right)^2 \\
&+2.\exp\left[-\frac{(1000-t_0)}{200}\right] - 2.\left(\exp\left[-\frac{(1000-t_0)}{200}\right]\right)^2 = 1 \\
&\Rightarrow Pc = 1
\end{aligned}$$

Thus we deduce that in the set C, we have a complete knowledge of the random variable since Pc = 1.

2.4.6. The Intersection Point:

$$\begin{aligned}
&P_r(t_0) = P_m(t_0)/i \Leftrightarrow \exp\left[-\frac{(1000-t_0)}{200}\right] \\
&= 1-\exp\left[-\frac{(1000-t_0)}{200}\right] \Leftrightarrow 2.\exp\left[-\frac{(1000-t_0)}{200}\right] = 1 \\
&\Leftrightarrow \exp\left[-\frac{(1000-t_0)}{200}\right] = \frac{1}{2} \Leftrightarrow \\
&t_0 = 1000 + 200.\text{Ln}(0.5) = 861.37 \\
&\text{And} \quad P_r(t_0 = 861.37) = \exp\left[-\frac{(1000-861.37)}{200}\right] = 0.5 \\
&\text{and} \quad P_m(t_0 = 861.37)/i = 1-\exp\left[-\frac{(1000-861.37)}{200}\right] \\
&= 1 - 0.5 = 0.5
\end{aligned}$$

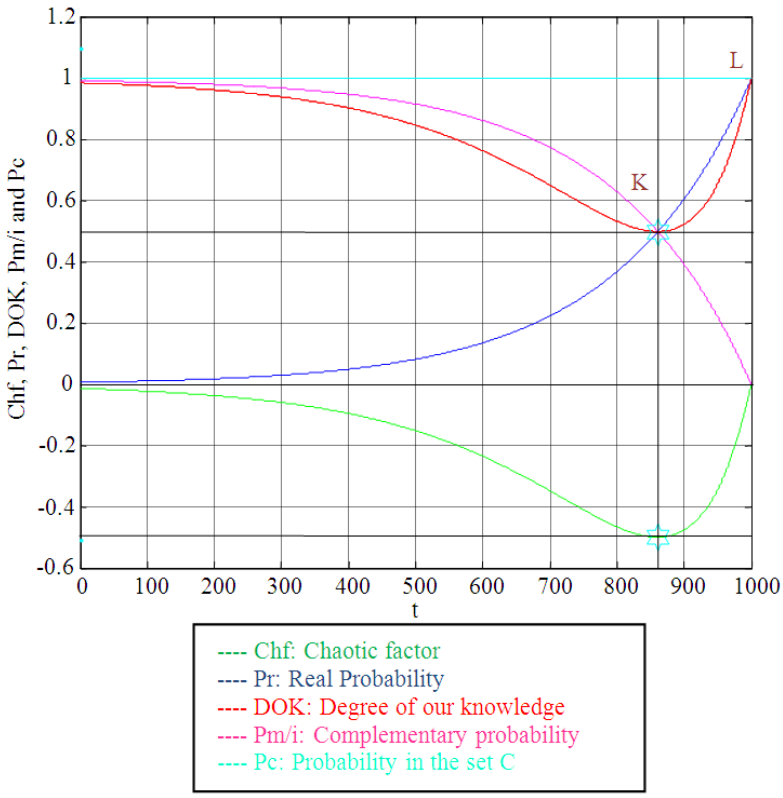

Fig. 17. EKA parameters in exponential cumulative distribution

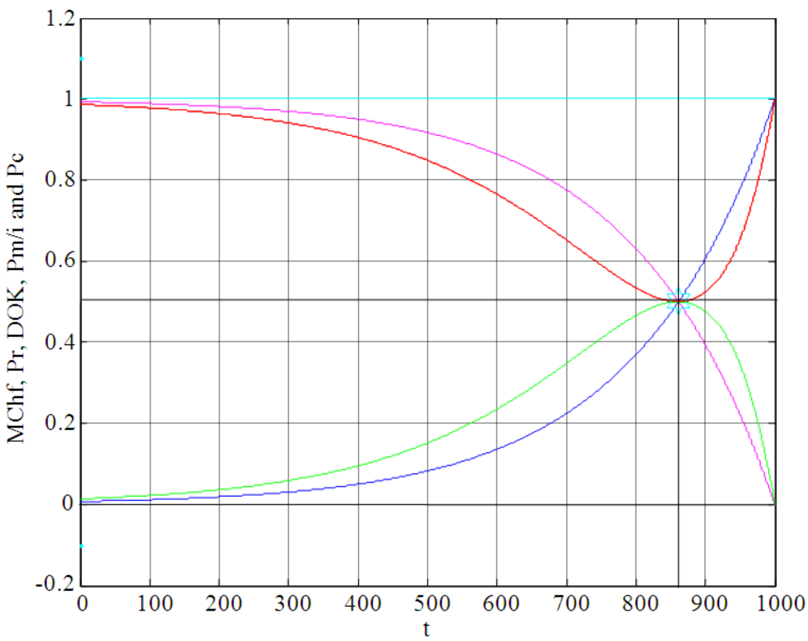

Fig. 18. EKA parameters in exponential cumulative distribution

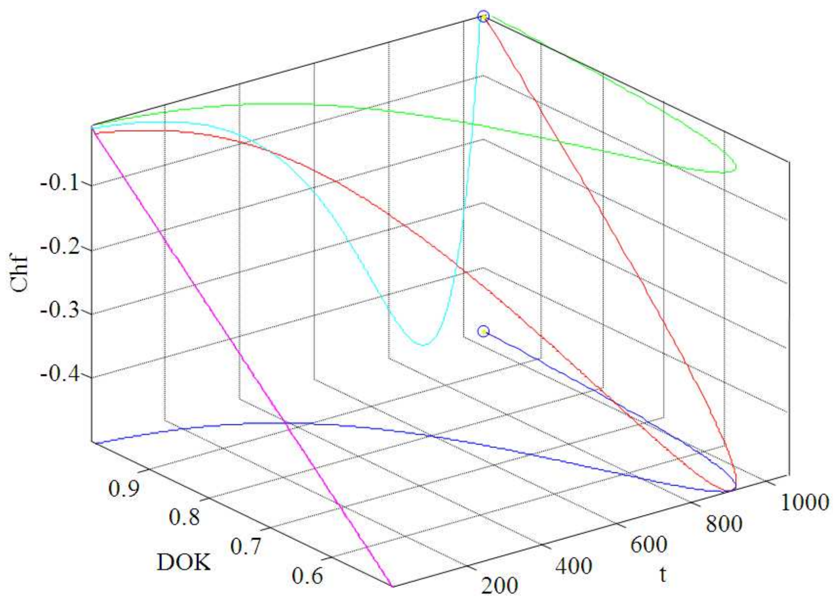

Fig. 19. DOK and Chf in terms of t and of each other in exponential cumulative distribution

So $P_r(t_0)$ and $P_m(t_0)/i$ intersect at (861.37, 0.5).

Moreover, the minimum of DOK and the maximum of MChf occur at (861.37, 0.5).

So we conclude that $P_r(t_0)$, $P_m(t_0)/i$, DOK and MChf all intersect at (861.37, 0.5) (**Fig. 17-19**).

2.4.7. The EKA Parameters Analysis in the Prognostic of Degradation:

In this case, we note from the figure above that the DOK is maximum (DOK=1) when MChf is minimum (MChf=0) (points J & L). Afterward, the magnitude of the chaotic factor MChf starts to increase with the decrease of DOK until it reaches 0.5 at t_0=861.37 (point K). Since the real probability P_r is an exponential distribution it will intersect with DOK at the point (861.37, 0.5) (point K). With the increase of t_0, Chf and MChf return to zero and the DOK returns to 1 where we reach total damage (D = 1) and hence the total certain failure (P_r = 1) of the system (point L). At this last point L the failure here is certain, $P_r(t_N) = t_N/t_N = 1$ and $RUL(t_N) = t_N - t_N = 0$ with $Pc(t_N) = 1$, so the logical explanation of the value DOK = 1 follows. We note that the point K is no more at the middle of DOK since the exponential distribution is not symmetric (**Fig. 17**).

2.5. The Normal Probability Distribution:

With a probability density function:

$$f(t) = \frac{dF(t)}{dt} = \frac{1}{\sqrt{2\pi}\sigma_t}\exp\left[-\frac{1}{2}\left(\frac{t-\bar{t}}{\sigma_t}\right)^2\right],$$

$$\text{for } -\infty < t < \infty$$

and a cumulative distribution function:

$$F(t_0) = \int_{-\infty}^{t_0} f(t)dt = \int_{0}^{t_0} f(t)dt = \int_{0}^{t_0} \frac{1}{\sqrt{2\pi}\sigma_t}\exp\left[-\frac{1}{2}\left(\frac{t-\bar{t}}{\sigma_t}\right)^2\right].dt,$$

$$\text{for } 0 \le t_0 \le t_N$$

We have taken the domain for the normal variable t_0 = [0, t_N = 1000], dt_0 = 0.1, $\bar{t} = 500$ (Mean) and σ_t = 150 (Standard deviation).

Note that:

$$\int_{-\infty}^{+\infty} dF = \int_{-\infty}^{+\infty} f(t)dt = \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}\sigma_t}\exp\left[-\frac{1}{2}\left(\frac{t-\bar{t}}{\sigma_t}\right)^2\right].dt = 1$$

2.5.1. The Real Probability P_r:

$$P_r(t_0) = F(t_0) = \int_{0}^{t_0} \frac{1}{\sqrt{2\pi}\sigma_t}\exp\left[-\frac{1}{2}\left(\frac{t-\bar{t}}{\sigma_t}\right)^2\right].dt$$

$$\text{if } 0 \le t_0 \le t_N = 1000$$

We note that $P_r(t_0)$ is a non-decreasing function.

2.5.2. The Complementary Probability P_m/i:

$$\begin{aligned}
&P_m(t_0)/i = 1 - P_r(t_0) = 1 - F(t_0)\\
&= 1 - \int_0^{t_0} \frac{1}{\sqrt{2\pi}\sigma_t} \exp\left[-\frac{1}{2}\left(\frac{t-\bar{t}}{\sigma_t}\right)^2\right].dt\\
&= \int_{t_0}^{t_N} \frac{1}{\sqrt{2\pi}\sigma_t} \exp\left[-\frac{1}{2}\left(\frac{t-\bar{t}}{\sigma_t}\right)^2\right].dt\\
&\text{if } 0 \le t_0 \le t_N = 1000
\end{aligned}$$

We note that $P_m(t_0)/i$ is a non-increasing function.

2.5.3. The Degree of Our Knowledge DOK:

DOK is the measure of our certain knowledge (100% probability) about the expected event, it does not include any uncertain knowledge (with probability less than 100%).

$$\begin{aligned}
&DOK(t_0) = Pc^2(t_0) + 2iP_r(t_0)P_m(t_0)\\
&= 1 - 2.P_{rob}(E).P_{rob}(\bar{E}) = 1 - 2.P_r(t_0).[1 - P_r(t_0)]\\
&= 1 - 2.F(t_0).[1 - F(t_0)] = 1 - 2.\int_0^{t_0} \frac{1}{\sqrt{2\pi}\sigma_t} \exp\left[-\frac{1}{2}\left(\frac{t-\bar{t}}{\sigma_t}\right)^2\right].dt\\
&\times \int_{t_0}^{t_N} \frac{1}{\sqrt{2\pi}\sigma_t} \exp\left[-\frac{1}{2}\left(\frac{t-\bar{t}}{\sigma_t}\right)^2\right].dt\\
&= 1 - 2.\int_0^{t_0} \frac{1}{\sqrt{2\pi}\sigma_t} \exp\left[-\frac{1}{2}\left(\frac{t-\bar{t}}{\sigma_t}\right)^2\right].dt\\
&+2.\left(\int_0^{t_0} \frac{1}{\sqrt{2\pi}\sigma_t} \exp\left[-\frac{1}{2}\left(\frac{t-\bar{t}}{\sigma_t}\right)^2\right].dt\right)^2
\end{aligned}$$

Which is a curve concave upward having a minimum at:

$$\begin{aligned}
&\int_0^{t_0} \frac{1}{\sqrt{2\pi}\sigma_t} \exp\left[-\frac{1}{2}\left(\frac{t-\bar{t}}{\sigma_t}\right)^2\right].dt\\
&= 0.5 \Leftrightarrow \text{ At } \left(t_0 = \bar{t} = 500,\ \ 0.5\right)
\end{aligned}$$

Since the normal distribution is symmetric about the mean which is $\bar{t} = 500$.

2.5.4. The Chaotic Factor Chf and MChf:

$$\begin{aligned}
&Chf(t_0) = 2iP_r(t_0)P_m(t_0) = -2.P_{rob}(E).P_{rob}(\bar{E})\\
&= -2.P_r(t_0).[1 - P_r(t_0)] = -2.F(t_0).[1 - F(t_0)]\\
&= -2.\int_0^{t_0} \frac{1}{\sqrt{2\pi}\sigma_t} \exp\left[-\frac{1}{2}\left(\frac{t-\bar{t}}{\sigma_t}\right)^2\right].dt\\
&\times \int_{t_0}^{t_N} \frac{1}{\sqrt{2\pi}\sigma_t} \exp\left[-\frac{1}{2}\left(\frac{t-\bar{t}}{\sigma_t}\right)^2\right].dt\\
&= -2.\int_0^{t_0} \frac{1}{\sqrt{2\pi}\sigma_t} \exp\left[-\frac{1}{2}\left(\frac{t-\bar{t}}{\sigma_t}\right)^2\right].dt\\
&+2.\left(\int_0^{t_0} \frac{1}{\sqrt{2\pi}\sigma_t} \exp\left[-\frac{1}{2}\left(\frac{t-\bar{t}}{\sigma_t}\right)^2\right].dt\right)^2
\end{aligned}$$

Which is a curve concave upward having a minimum at:

$$\begin{aligned}
&\int_0^{t_0} \frac{1}{\sqrt{2\pi}\sigma_t} \exp\left[-\frac{1}{2}\left(\frac{t-\bar{t}}{\sigma_t}\right)^2\right].dt\\
&= 0.5 \Leftrightarrow \text{ At } \left(t_0 = \bar{t} = 500,\ \ -0.5\right)
\end{aligned}$$

Since the normal distribution is symmetric about the mean which is $\bar{t} = 500$.

Therefore, we can infer the magnitude of the chaotic factor MChf:

$$\begin{aligned}
&MChf(t_0) = |Chf(t_0)| = |-2.P_r(t_0).[1 - P_r(t_0)]|\\
&= 2.P_r(t_0).[1 - P_r(t_0)] = 2.F(t_0).[1 - F(t_0)]\\
&= 2.\int_0^{t_0} \frac{1}{\sqrt{2\pi}\sigma_t} \exp\left[-\frac{1}{2}\left(\frac{t-\bar{t}}{\sigma_t}\right)^2\right].dt\\
&\times \int_{t_0}^{t_N} \frac{1}{\sqrt{2\pi}\sigma_t} \exp\left[-\frac{1}{2}\left(\frac{t-\bar{t}}{\sigma_t}\right)^2\right].dt\\
&= 2.\int_0^{t_0} \frac{1}{\sqrt{2\pi}\sigma_t} \exp\left[-\frac{1}{2}\left(\frac{t-\bar{t}}{\sigma_t}\right)^2\right].dt\\
&-2.\left(\int_0^{t_0} \frac{1}{\sqrt{2\pi}\sigma_t} \exp\left[-\frac{1}{2}\left(\frac{t-\bar{t}}{\sigma_t}\right)^2\right].dt\right)^2
\end{aligned}$$

Which is a curve concave downward having a maximum at:

$$\begin{aligned}
&\int_0^{t_0} \frac{1}{\sqrt{2\pi}\sigma_t} \exp\left[-\frac{1}{2}\left(\frac{t-\bar{t}}{\sigma_t}\right)^2\right].dt\\
&= 0.5 \Leftrightarrow \text{ At } \left(t_0 = \bar{t} = 500,\ \ 0.5\right)
\end{aligned}$$

Since the normal distribution is symmetric about the mean which is $\bar{t} = 500$.

2.5.5. Pc: The Probability in the Complex Set C:

$$\begin{aligned}
&Pc^2(t_0) = DOK(t_0) - Chf(t_0)\\
&= 1 - 2.\int_0^{t_0} \frac{1}{\sqrt{2\pi}\sigma_t} \exp\left[-\frac{1}{2}\left(\frac{t-\bar{t}}{\sigma_t}\right)^2\right].dt\\
&\times \int_{t_0}^{t_N} \frac{1}{\sqrt{2\pi}\sigma_t} \exp\left[-\frac{1}{2}\left(\frac{t-\bar{t}}{\sigma_t}\right)^2\right].dt\\
&+2.\int_0^{t_0} \frac{1}{\sqrt{2\pi}\sigma_t} \exp\left[-\frac{1}{2}\left(\frac{t-\bar{t}}{\sigma_t}\right)^2\right].dt\\
&\times \int_{t_0}^{t_N} \frac{1}{\sqrt{2\pi}\sigma_t} \exp\left[-\frac{1}{2}\left(\frac{t-\bar{t}}{\sigma_t}\right)^2\right].dt = 1\\
&\Rightarrow Pc(t_0) = 1
\end{aligned}$$

Thus we deduce that in the set C, we have a complete knowledge of the random variable since Pc = 1.

2.5.6. The Intersection Point:

$$\begin{aligned}
&P_r(t_0) = P_m(t_0)/i \Leftrightarrow \int_0^{t_0} \frac{1}{\sqrt{2\pi}\sigma_t} \exp\left[-\frac{1}{2}\left(\frac{t-\bar{t}}{\sigma_t}\right)^2\right].dt\\
&= 1 - \int_0^{t_0} \frac{1}{\sqrt{2\pi}\sigma_t} \exp\left[-\frac{1}{2}\left(\frac{t-\bar{t}}{\sigma_t}\right)^2\right].dt\\
&\Leftrightarrow 2.\int_0^{t_0} \frac{1}{\sqrt{2\pi}\sigma_t} \exp\left[-\frac{1}{2}\left(\frac{t-\bar{t}}{\sigma_t}\right)^2\right].dt = 1\\
&\Leftrightarrow \int_0^{t_0} \frac{1}{\sqrt{2\pi}\sigma_t} \exp\left[-\frac{1}{2}\left(\frac{t-\bar{t}}{\sigma_t}\right)^2\right].dt, = \frac{1}{2} = 0.5\\
&\Leftrightarrow t_0 = \bar{t} = 500\\
&\text{and} \quad P_r(t_0 = \bar{t} = 500) = 0.5\\
&\text{and} \quad P_m(t_0 = \bar{t} = 500)/i = 1 - 0.5 = 0.5
\end{aligned}$$

So $P_r(t_0)$ and $P_m(t_0)/i$ intersect at (500, 0.5).

Moreover, the minimum of DOK and the maximum of MChf occur at (500, 0.5).

So we conclude that $P_r(t_0)$, $P_m(t_0)/i$, DOK and MChf all intersect at (500, 0.5) (**Fig. 20-22**).

2.5.7. The EKA Parameters Analysis in the Prognostic of Degradation:

In this case, we note from the figure that the DOK is maximum (DOK = 1) when MChf is minimum (MChf = 0) (points J & L). Afterward, the magnitude of the chaotic factor MChf starts to increase with the decrease of DOK until it reaches 0.5 at $t_0 = 500$ (point K). Since the real probability P_r is a normal distribution it will intersect with DOK at the point (500, 0.5) (point K). With the increase of t_0, Chf and MChf return to zero and the DOK returns to 1 where we reach total damage (D = 1) and hence the total certain failure ($P_r = 1$) of the system (point L). At this last point L the failure here is certain, $P_r(t_N) = t_N/t_N = 1$ and $RUL(t_N) = t_N\text{-}t_N = 0$ with $Pc(t_N) = 1$, so the logical explanation of the value DOK=1 follows. We note that the point K is at the middle of DOK since the normal distribution is symmetric (**Fig. 20**).

2.6. The Lognormal Probability Distribution:

With a probability density function:

$$f(t) = \frac{dF(t)}{dt} = \frac{1}{\sqrt{2\pi}\sigma_t t} \exp\left[-\frac{1}{2}\left(\frac{Ln(t)-\bar{t}}{\sigma_t}\right)^2\right], \quad \text{for} \quad 0 < t < \infty$$

and a cumulative distribution function:

$$F(t_0) = \int_0^{t_0} f(t)dt = \int_0^{t_0} \frac{1}{\sqrt{2\pi}\sigma_t t} \exp\left[-\frac{1}{2}\left(\frac{Ln(t)-\bar{t}}{\sigma_t}\right)^2\right] dt \quad \text{for } 0 \le t_0 \le t_N$$

We have taken the domain for the lognormal variable $t_0 = [0, t_N = 1000]$, $dt_0 = 0.1$, $\bar{t} = 5.3$ (mean) and $\sigma_t = 0.7$ (standard deviation).

Note that:

$$\int_0^{+\infty} dF = \int_0^{+\infty} f(t)dt = \int_0^{+\infty} \frac{1}{\sqrt{2\pi}\sigma_t t} \exp\left[-\frac{1}{2}\left(\frac{Ln(t)-\bar{t}}{\sigma_t}\right)^2\right].dt = 1$$

2.6.1. The Real Probability P_r:

$$P_r(t_0) = F(t_0) = \int_0^{t_0} \frac{1}{\sqrt{2\pi}\sigma_t t} \exp\left[-\frac{1}{2}\left(\frac{Ln(t)-\bar{t}}{\sigma_t}\right)^2\right].dt \quad \text{if } 0 \le t_0 \le t_N = 1000$$

We note that $P_r(t_0)$ is a non-decreasing function.

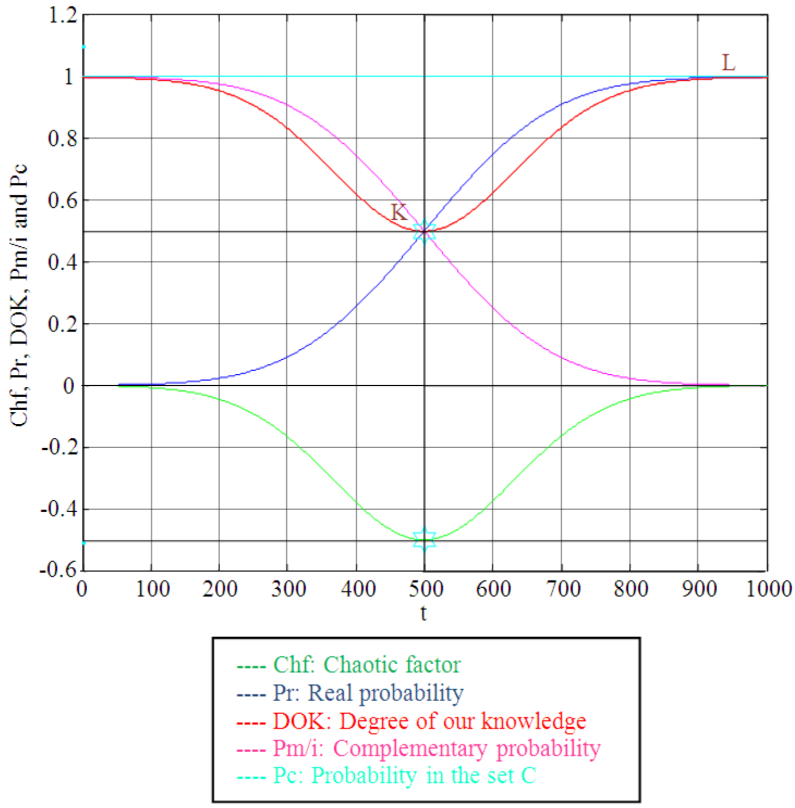

Fig. 20. EKA parameters in normal probability distribution

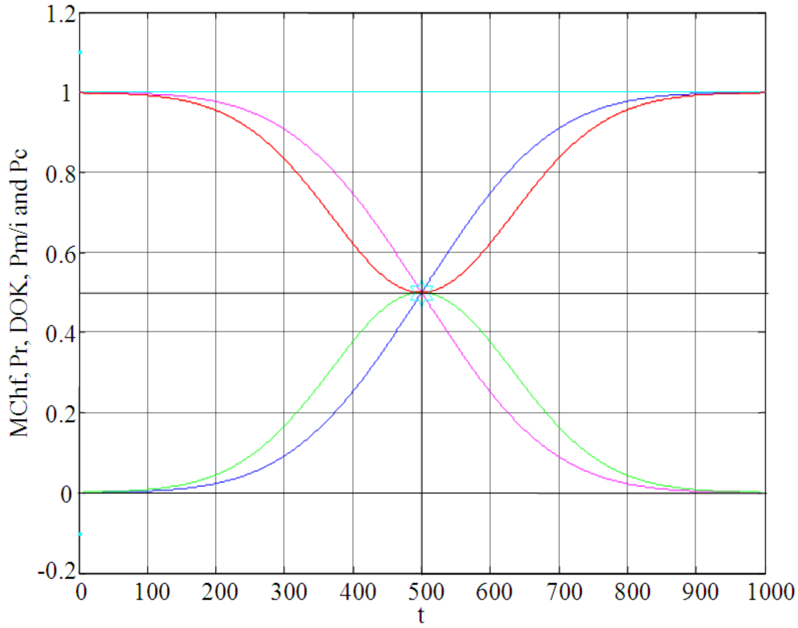

Fig. 21. EKA parameters in normal probability distribution

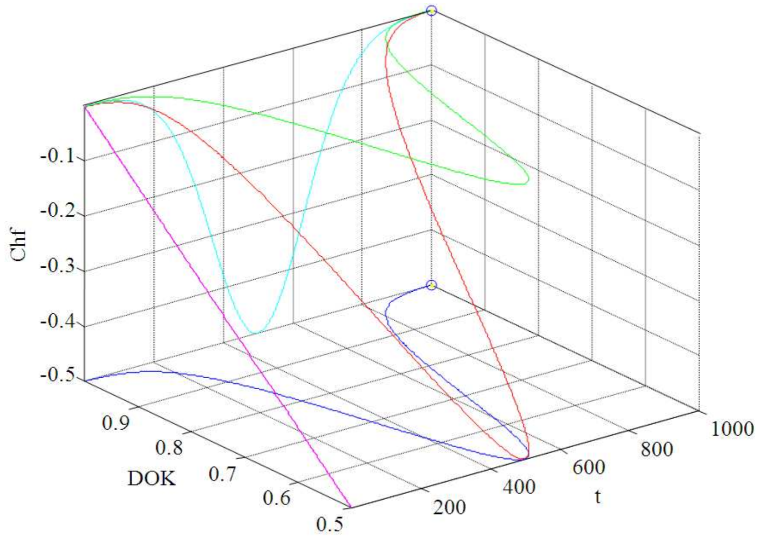

Fig. 22. DOK and Chf in terms of t and of each other in normal probability distribution

2.6.2. The Complementary Probability P_m/i:

$$P_m(t_0)/i = 1 - P_r(t_0) = 1 - F(t_0)$$
$$= 1 - \int_0^{t_0} \frac{1}{\sqrt{2\pi}\sigma_t t} \exp\left[-\frac{1}{2}\left(\frac{Ln(t)-\bar{t}}{\sigma_t}\right)^2\right].dt$$
$$= \int_{t_0}^{t_N} \frac{1}{\sqrt{2\pi}\sigma_t t} \exp\left[-\frac{1}{2}\left(\frac{Ln(t)-\bar{t}}{\sigma_t}\right)^2\right].dt$$
$$\text{if } 0 \le t_0 \le t_N = 1000$$

We note that $P_m(t_0)/i$ is a non-increasing function.

2.6.3. The Degree of Our Knowledge DOK:

DOK is the measure of our certain knowledge (100% probability) about the expected event, it does not include any uncertain knowledge (with probability less than 100 %):

$$DOK(t_0) = Pc^2(t_0) + 2iP_r(t_0)P_m(t_0)$$
$$= 1 - 2.P_{rob}(E).P_{rob}(\bar{E}) = 1 - 2.P_r(t_0).[1 - P_r(t_0)]$$
$$= 1 - 2.F(t_0).[1 - F(t_0)]$$
$$= 1 - 2.\int_0^{t_0} \frac{1}{\sqrt{2\pi}\sigma_t t} \exp\left[-\frac{1}{2}\left(\frac{Ln(t)-\bar{t}}{\sigma_t}\right)^2\right].dt$$
$$\times \int_{t_0}^{t_N} \frac{1}{\sqrt{2\pi}\sigma_t t} \exp\left[-\frac{1}{2}\left(\frac{Ln(t)-\bar{t}}{\sigma_t}\right)^2\right].dt$$
$$= 1 - 2.\int_0^{t_0} \frac{1}{\sqrt{2\pi}\sigma_t t} \exp\left[-\frac{1}{2}\left(\frac{Ln(t)-\bar{t}}{\sigma_t}\right)^2\right].dt$$
$$+2.\left(\int_0^{t_0} \frac{1}{\sqrt{2\pi}\sigma_t t} \exp\left[-\frac{1}{2}\left(\frac{Ln(t)-\bar{t}}{\sigma_t}\right)^2\right].dt\right)^2$$

which is a curve concave upward having a minimum at:

$$\int_0^{t_0} \frac{1}{\sqrt{2\pi}\sigma_t t} \exp\left[-\frac{1}{2}\left(\frac{Ln(t)-\bar{t}}{\sigma_t}\right)^2\right].dt$$
$$= 0.5 \Leftrightarrow \text{ At } (t_0 = 200.34, \ 0.5)$$

Notice that $Ln(200.34) = 5.3 = \bar{t}$ which is the mean of the distribution and equivalently $\exp(\bar{t} = 5.3) = 200.34$. This is conformed to the lognormal distribution.

2.6.4. The Chaotic Factor Chf and MChf:

$$Chf(t_0) = 2iP_r(t_0)P_m(t_0) = -2.P_{rob}(E).P_{rob}(\bar{E})$$
$$= -2.P_r(t_0).[1 - P_r(t_0)] = -2.F(t_0).[1 - F(t_0)]$$
$$= -2.\int_0^{t_0} \frac{1}{\sqrt{2\pi}\sigma_t t} \exp\left[-\frac{1}{2}\left(\frac{Ln(t)-\bar{t}}{\sigma_t}\right)^2\right].dt$$
$$\times \int_{t_0}^{t_N} \frac{1}{\sqrt{2\pi}\sigma_t t} \exp\left[-\frac{1}{2}\left(\frac{Ln(t)-\bar{t}}{\sigma_t}\right)^2\right].dt$$
$$= -2.\int_0^{t_0} \frac{1}{\sqrt{2\pi}\sigma_t t} \exp\left[-\frac{1}{2}\left(\frac{Ln(t)-\bar{t}}{\sigma_t}\right)^2\right].dt$$
$$+2.\left(\int_0^{t_0} \frac{1}{\sqrt{2\pi}\sigma_t t} \exp\left[-\frac{1}{2}\left(\frac{Ln(t)-\bar{t}}{\sigma_t}\right)^2\right].dt\right)^2$$

which is a curve concave upward having a minimum at:

$$\int_0^{t_0} \frac{1}{\sqrt{2\pi}\sigma_t t} \exp\left[-\frac{1}{2}\left(\frac{Ln(t)-\bar{t}}{\sigma_t}\right)^2\right].dt = 0.5$$
$$\Leftrightarrow \text{ At } \left(t_0 = \exp\left(\bar{t} = 5.3\right) = 200.34, \quad -0.5\right)$$

Therefore, we can infer the magnitude of the chaotic factor MChf:

$$\begin{aligned}
&MChf(t_0) = |Chf(t_0)| = |-2.P_r(t_0).[1-P_r(t_0)]| \\
&= 2.P_r(t_0).[1-P_r(t_0)] = 2.F(t_0).[1-F(t_0)] \\
&= 2.\int_0^{t_0} \frac{1}{\sqrt{2\pi}\sigma_t t} \exp\left[-\frac{1}{2}\left(\frac{Ln(t)-\bar{t}}{\sigma_t}\right)^2\right].dt \\
&\times \int_{t_0}^{t_N} \frac{1}{\sqrt{2\pi}\sigma_t t} \exp\left[-\frac{1}{2}\left(\frac{Ln(t)-\bar{t}}{\sigma_t}\right)^2\right].dt \\
&= 2.\int_0^{t_0} \frac{1}{\sqrt{2\pi}\sigma_t t} \exp\left[-\frac{1}{2}\left(\frac{Ln(t)-\bar{t}}{\sigma_t}\right)^2\right].dt \\
&-2.\left(\int_0^{t_0} \frac{1}{\sqrt{2\pi}\sigma_t t} \exp\left[-\frac{1}{2}\left(\frac{Ln(t)-\bar{t}}{\sigma_t}\right)^2\right].dt\right)^2
\end{aligned}$$

which is a curve concave downward having a maximum at:

$$\int_0^{t_0} \frac{1}{\sqrt{2\pi}\sigma_t t} \exp\left[-\frac{1}{2}\left(\frac{Ln(t)-\bar{t}}{\sigma_t}\right)^2\right].dt = 0.5$$
$$\Leftrightarrow \text{ At } \left(t_0 = \exp\left(\bar{t} = 5.3\right) = 200.34, \quad 0.5\right)$$

2.6.5. Pc: Probability in the Complex Set C:

$$\begin{aligned}
&Pc^2(t_0) = DOK(t_0) - Chf(t_0) \\
&= 1 - 2.\int_0^{t_0} \frac{1}{\sqrt{2\pi}\sigma_t t} \exp\left[-\frac{1}{2}\left(\frac{Ln(t)-\bar{t}}{\sigma_t}\right)^2\right].dt \\
&\times \int_{t_0}^{t_N} \frac{1}{\sqrt{2\pi}\sigma_t t} \exp\left[-\frac{1}{2}\left(\frac{Ln(t)-\bar{t}}{\sigma_t}\right)^2\right].dt \\
&+2.\int_0^{t_0} \frac{1}{\sqrt{2\pi}\sigma_t t} \exp\left[-\frac{1}{2}\left(\frac{Ln(t)-\bar{t}}{\sigma_t}\right)^2\right].dt \\
&\times \int_{t_0}^{t_N} \frac{1}{\sqrt{2\pi}\sigma_t t} \exp\left[-\frac{1}{2}\left(\frac{Ln(t)-\bar{t}}{\sigma_t}\right)^2\right].dt = 1 \Rightarrow Pc(t_0) = 1
\end{aligned}$$

Thus we deduce that in the set C, we have a complete knowledge of the random variable since Pc = 1.

2.6.6. The Intersection Point:

$$\begin{aligned}
&P_r(t_0) = P_m(t_0)/i \Leftrightarrow \int_0^{t_0} \frac{1}{\sqrt{2\pi}\sigma_t t} \exp\left[-\frac{1}{2}\left(\frac{Ln(t)-\bar{t}}{\sigma_t}\right)^2\right].dt \\
&= 1 - \int_0^{t_0} \frac{1}{\sqrt{2\pi}\sigma_t t} \exp\left[-\frac{1}{2}\left(\frac{Ln(t)-\bar{t}}{\sigma_t}\right)^2\right].dt \\
&\Leftrightarrow 2.\int_0^{t_0} \frac{1}{\sqrt{2\pi}\sigma_t t} \exp\left[-\frac{1}{2}\left(\frac{Ln(t)-\bar{t}}{\sigma_t}\right)^2\right].dt = 1 \\
&\Leftrightarrow \int_0^{t_0} \frac{1}{\sqrt{2\pi}\sigma_t t} \exp\left[-\frac{1}{2}\left(\frac{Ln(t)-\bar{t}}{\sigma_t}\right)^2\right].dt = \frac{1}{2} = 0.5 \\
&\Leftrightarrow t_0 = 200.34 = \exp\left(\bar{t} = 5.3\right) \\
&\qquad \text{And } \ P_r(t_0 = 200.34) = 0.5 \\
&\qquad \text{and } \ P_m(t_0 = 200.34)/i = 1 - 0.5 = 0.5
\end{aligned}$$

So $P_r(t_0)$ and $P_m(t_0)/i$ intersect at (200.34, 0.5).

Moreover, the minimum of DOK and the maximum of MChf occur at (200.34, 0.5).

So we conclude that $P_r(t_0)$, $P_m(t_0)/i$, DOK and MChf all intersect at (200.34, 0.5) (**Fig. 23-25**).

2.6.7. The EKA Parameters Analysis in the Prognostic of Degradation:

In this case, we note from the figure below that the DOK is maximum (DOK=1) when MChf is minimum (MChf = 0) (points J & L). Afterward, the magnitude of the chaotic factor MChf starts to increase with the decrease of DOK until it reaches 0.5 at t_0 = 200.34 (point K). Since the real probability P_r is a lognormal distribution it will intersect with DOK at the point (200.34, 0.5) (point K). With the increase of t_0, Chf and MChf return to zero and the DOK returns to 1 where we reach total damage (D = 1) and hence the total certain failure (P_r = 1) of the system (point L). We note that the point K is no more at the middle of DOK since the lognormal distribution is not symmetric.

At each instant t_0, the remaining useful lifetime RUL(t_0) is certainly predicted in the complex set C with Pc maintained as equal to one through a continuous compensation between DOK and Chf. This compensation is from instant t_0 = 0 where $D(t_0)$ = 0 until the failure instant t_N where $D(t_N)$ = 1 (**Fig. 23**).

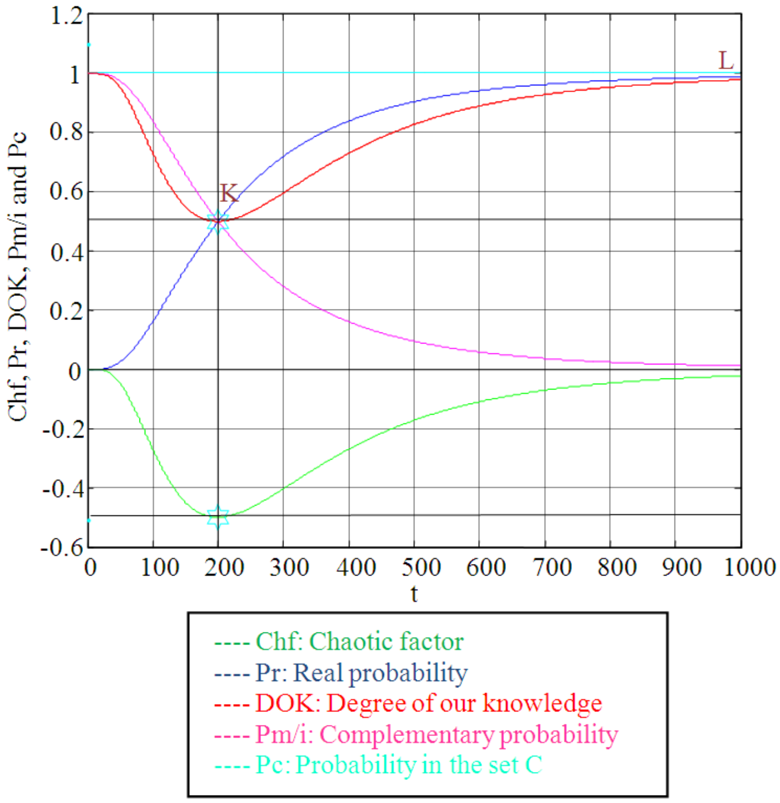

Fig. 23. EKA parameters in log-normal probability distribution

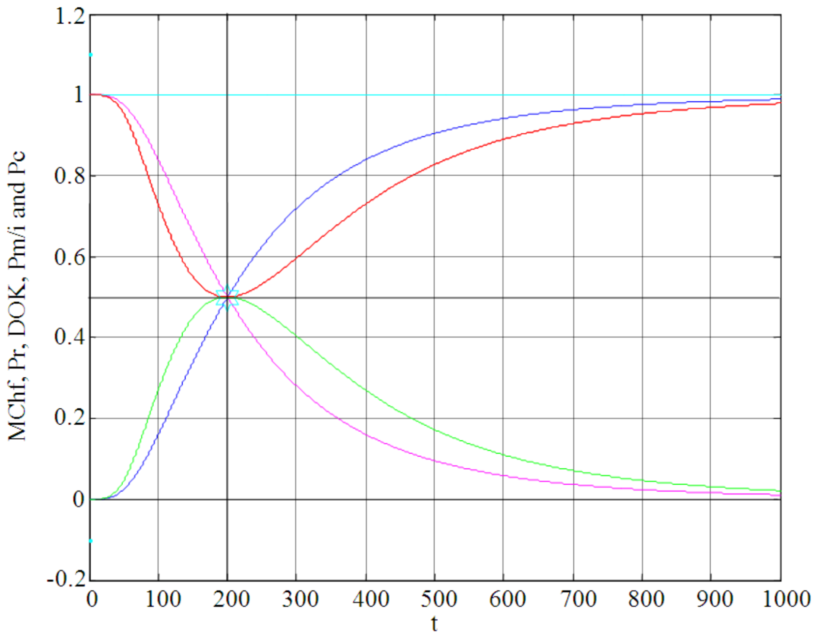

Fig. 24. EKA parameters in log-normal probability distribution

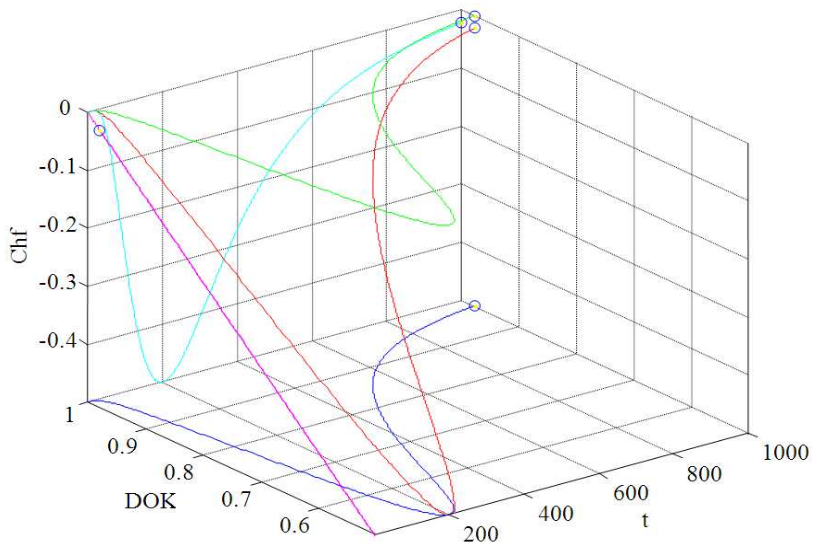

Fig. 25. DOK and Chf in terms of t and of each other in log-normal probability distribution

3. CONCLUSION

In this study I applied the theory of Extended Kolmogorov Axioms to different probability distributions: the uniform, the logarithmic, the power, the exponential, the normal and the lognormal cumulative probability distributions. In addition, I established a tight link between the new theory and degradation or the remaining useful lifetime. Hence, I developed the theory of "Complex Probability" beyond the scope of the previous first and second paper on this topic. As it was proved and illustrated, when the degradation index is 0 or 1 and correspondingly the RUL is t_N or 0 then the Degree of Our Knowledge (DOK) is one and the chaotic factor (Chf and MChf) is 0 since the state of the system is totally known. During the process of degradation ($0<D<1$) we have: $0.5<DOK<1$, $-0.5<Chf<0$ and $0<MChf<0.5$. Notice that during the whole process of degradation we have $Pc = DOK - Chf = DOK + MChf = 1$, that means that the phenomenon which seems to be random and stochastic in R is now deterministic and certain in $C = R + M$ and this after adding to R the contributions of M and hence after subtracting the chaotic factor from the degree of our knowledge. Moreover, for each value of an instant t_0, I have determined its corresponding probability of survival or of the remaining useful lifetime $RUL(t_0) = t_N - t_0$. In other words, at each instant t_0, $RUL(t_0)$ is certainly predicted in the complex set C with Pc maintained as equal to one through a continuous compensation between DOK and Chf. This compensation is from instant $t_0 = 0$ where $D(t_0) = 0$ until the failure instant t_N where $D(t_N) = 1$. Furthermore, using all these graphs illustrated throughout the whole paper, we can visualize and quantify both the system chaos (Chf and MChf) and the system certain knowledge (DOK and Pc). This is certainly very interesting and fruitful and shows once again the benefits of extending Kolmogorov's axioms and thus the originality and usefulness of this new field in mathematics that can be called verily: "The Complex Probability and Statistics Paradigm".

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