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Combinatorial Properties of Modified Chordal Rings Degree Four Networks

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Abstract: Problem statement: Modified Chordal Rings Degree Four, called CHRm4 is the first modified structure of chordal rings. This CHRm4 is an undirected circulant graph and is a double loop graph. **Approach:** This study presented the main properties of CHRm4. There are connectivity, Hamiltonian cycle and asymmetric. **Results:** Several definitions, postulates, corollary, theorems and lemmas were constructed according to these three main properties. It is about interconnection between nodes, how the Hamiltonian cycle was occurred and why CHRm4 is not symmetric. **Conclusion:** From these three properties, there are two dominant properties obviously shown that the CHRm4 is contrary from the previous traditional Chordal Rings Degree Four (CR4). It is connectivity and asymmetric. There are different connections for odd and even nodes, therefore CHRm4 is not symmetric. The Hamiltonian cycle property has the same concept with CR4.

Key words: Circulant graphs, interconnection network, connectivity, Hamiltonian cycle, asymmetric

INTRODUCTION

Interconnection networks have been used in the design of Local Area Networks (LAN), telecommunication networks and other distributed computer systems. Rings technology previously has a poor performance compare to star, torus, mesh, complete graph and bus. These problems can be solved by adding more links or chord lengths into the ring in a uniform way to form chordal rings. It is desirable to add a few links as possible. Mostly topological characteristics determined the performance and robustness of any networks. In chordal rings, vertices were representing their processing elements and edges representing the communication links between them. Arden and Lee (1981) were introduced the first chordal rings of degree three by adding a chord into the ring. Chordal ring is a circulant graph.

According to Liestman *et al*. (1998) and Dubalski *et al*. (2007), circulant graphs have deserved significant attention, the traditional ring and the complete graph topologies belong to this class of graphs. Circulants of different degrees constituted the basis of some classical distributed and parallel systems. The design of certain data alignment networks for complex memory systems have also relied on circulant graphs.

A circulant graph with N vertex and chord lengths $\{s_1,s_2,...,s_m\}$ is an undirected graph in which each vertex n, $0 \le n \le N-1$ is adjacent to all the vertex $n \pm s_i \bmod N$ with $1 \le i \le m$ and this graph denoted as $C_N(s_1,s_2,\ldots,s_m)$. The family of circulant graphs includes the complete graph and the cyclic graph or ring among its members.

Background: Network topologies based on traditional chordal rings network were discussed before are Chordal Rings Degree Three (CR3) (Barriere, 2003), Chordal Rings Degree Four (CR4) and Chordal Rings Degree Six (CR6) (Farah Azura *et al*., 2008; Farah *et al*., 2008). Arden and Lee (1981) had introduced CR3. CR4 was discussed by (Narayanan and Opatrny, 1999; Narayanan *et al*., 2001; Browne and Hodgson, 1990; Bujnowski *et al*., 2004) and CR6 was discussed by Abbas and Othman (2007). Modified Degree Four Chordal Rings (CHRm) was introduced by (Dubalski *et al*., 2007) and the properties of this CHRm were not discussed on details before.

MATERIALS AND METHODS

Figure 1a shows a graph of CR3(16;1,3). Arden and Lee (1981) are assumed the number of nodes, n is

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even and nodes are indexed 0,1,2,…,(n-1) around the ring. In the present correspondence they restrict their attention to the case in which each odd-numbered node i(i = 1,3,…,n-2) is connected to a node (i+w)mod n. Each even-numbered node j(j = 0,2,…,n-2) is connected to a node (j-w) mod n. The chord length, w is assumed to be positive odd and they assumed that $w \le \frac{n}{2}$.

CR4 is defined as chordal rings network in which each node has two circumferential and two chordal links (Browne and Hodgson, 1990). Figure 1b shows a graph of CR4(16;1,3). The ring contains N nodes (processors). These nodes will be assumed to be indexed in ascending order for clockwise direction as 0,1,…,(N-1) and -1,-2,…, so that -1 is the same as (N-1) for anticlockwise direction. i is connected to nodes (i-1) and (i+l), or more briefly nodes i±1. In particular node 0 is connected to node 1 and (N-1).

CR6 is related to a distributed loop graph $G(n;s_1;s_2;,,,s_d)$ which is a graph with a vertex set equal to 0,1,…,n-1 and the edge set equal to (u, u±i) where $0<u<n-1, i \in \{s_1;s_2;,,,s_d\}$ CR6 was denoted by $G(n;s_1;s_2)$ where n as the number of nodes and s_1, s_2 the length of the chords. Figure 1c shows a graph of CR6(16;3;6).

CHRm4 is adaptation from CHRm which consists of one ring with p nodes, where p is even. For $0 \le k \le \frac{p}{2}$, each even node i_{2k} is additionally connected to two nodes $i_{(2k-q_1)(\text{mod } p)}$ and $i_{(2k+q_1)(\text{mod } p)}$, while each odd node $i_{(2k+1)}$ is connected to two nodes $i_{(2k+1-q_2)(\text{mod } p)}$ and $i_{(2k+1+q_2)(\text{mod } p)}$ by chords of even lengths q_1 and q_2, not exceeding $\frac{p}{2}$. The graph CHRm is denoted as $(p;q_1/q_2)$ (Dubalski *et al*., 2007). Figure 1d shows a graph of CHRm4 (16,1,4,6) with two chord lengths.

According to Arden and Lee (1981), the performance of chordal rings can be improved by increasing the number of nodes, decreasing the diameter and increasing the node degree.

There are many criteria for a comparison of interconnection topologies which have been proposed such as degree, number of nodes, diameter, average path lengths, connectivity, Hamiltonian cycle, symmetric, isomorphism, regularity and reliability. In this study, we investigate the network properties of CHRm4. This CHRm4 is a type of circulant graph. In this research, we will show that this type of circulant graphs have several properties such as connectivity, Hamiltonian cycle and asymmetric.

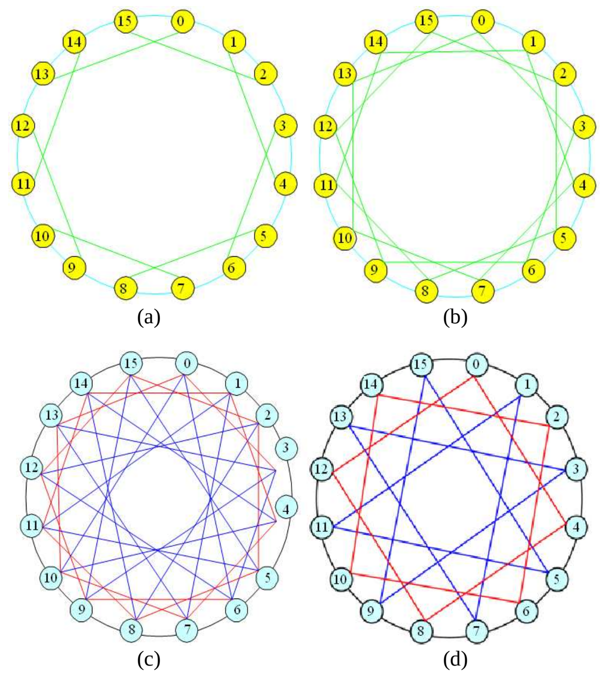


Fig. 1: Chordal rings network graphs. (a): CR3(16;1,3); (b) CR4(16;1,3); (c): CR6(16;3;6); (d) CHRm4 (16,1,4,6)

Properties:

Connectivity: Let the connectivity is between source node, i to destination node, j. The following Definition 1 is adapted from Dubalski *et al*. (2007) and it is describes how to construct the structure of CHRm4.

Definition 1: The modified chordal rings degree four, CHRm4 consists of one ring of N nodes, where N is positive and even number of nodes. This graph is denoted as $CHm4(N, s, h_1, h_2)$. The values of ring edges, s must be 1 and the values of chord lengths, $h_1$and h_2 must be positive and even with $h_1, h_2 \in$ h. For 0≤k<N/2, each even-numbered node, i_{2k} = {0,2,…,N-2} is additionally connected to $i_{(2k+1-h_2)(\text{mod } N)}$ and $i_{(2k-h_1)(\text{mod } N)}$, while odd-numbered node, i_{2k+1} = (1,3,…,N-1) is additionally connected to $i_{(2k+1+h_2)(\text{mod } N)}$ and $i_{(2k+1-h_2)(\text{mod } N)}$. The values of N and h_1 and also for N and h_2 for CHRm4 must have $gcd(N,h_1,h_2) = 2$.

The following Postulate 1 describes the distance function of $CHRm4(N, s, h_1, h_2)$ and fulfills the Definition 1.

Postulate 1: A CHRm4 obeys all distance functions of:

- $d(i, j) \ge 0$ and $d(i,j) = 0 \leftrightarrow i = j$

- d(i, j) = d(j, i) for all i, j
- d(i, k)≤d(i, j)+d(j,k) for all i, j, k

The following Theorem 1 describe about the connectivity in CHRm4 among all nodes.

Theorem 1: CHRm4(N, s, h_1, h_2) is all connected if and only if gcd(N, s, h_1, h_2) = 1.

Proof:

Necessity: If gcd(N, s, h_1, h_2) = d<1, then node i_{2k} can only reach node $i_{(2k\pm h_1)}$ if $i_{2k} \equiv i_{(2k\pm h_1)(\text{mod d})}$ and node i_{2k+1} can only reach $i_{(2k+1\pm h_2)}$ if $i_{2k+1} \equiv i_{(2k+1\pm h_2)(\text{mod d})}$ where $0 \le k < \frac{N}{2}$.

Sufficiency: Suppose gcd(N, s, h_1, h_2) = 1. Then there exist p, q and r such that ps+qh_1+rh2 = ∝ where ∝ = gcd(s, h_1, h_2). Since gcd(N, s, h_1, h_2) = 1, i.e., gcd(N, a) = 1, there exist β such that β(ps+ qh_1+rh2) = α, β ≡ 1(mod N). Let

- |βp|, s-steps ((-s)-steps if p<0)
- |βq|,h_1-steps((-h_1)-steps if q<0)
- |βr|, h_2-steps((-h_2)-steps if r<0)

Every even node, i_{2k} and odd node, i_{2k+1} was connected by induction.

Hamiltonian cycle: A Hamiltonian cycle is a cycle in an undirected graph which visits each vertex exactly once and also returns to the starting vertex. CHRm4 contains Hamiltonian cycle and it has no cut-vertices. Lemma 1 proved how the Hamiltonian cycle was completed.

Lemma 1: A Hamiltonian cycle of CHRm4 cannot contain:

- $\{\overrightarrow{h_1}, \overleftarrow{h_1}\}$
- $\{-\overrightarrow{h_1}, -\overleftarrow{h_1}\}$
- $\{\overrightarrow{h_2}, \overleftarrow{h_2}\}$
- $\{-\overrightarrow{h_2}, -\overleftarrow{h_2}\}$

Proof: The nodes in Fig. 2 represented as Triangular Prism for Chords-Ring Edges. s represent ring edges, h_1 represent a chord length for even nodes and h_2 represent a chord length for odd nodes. For $0 \le k < \frac{N}{2}$,

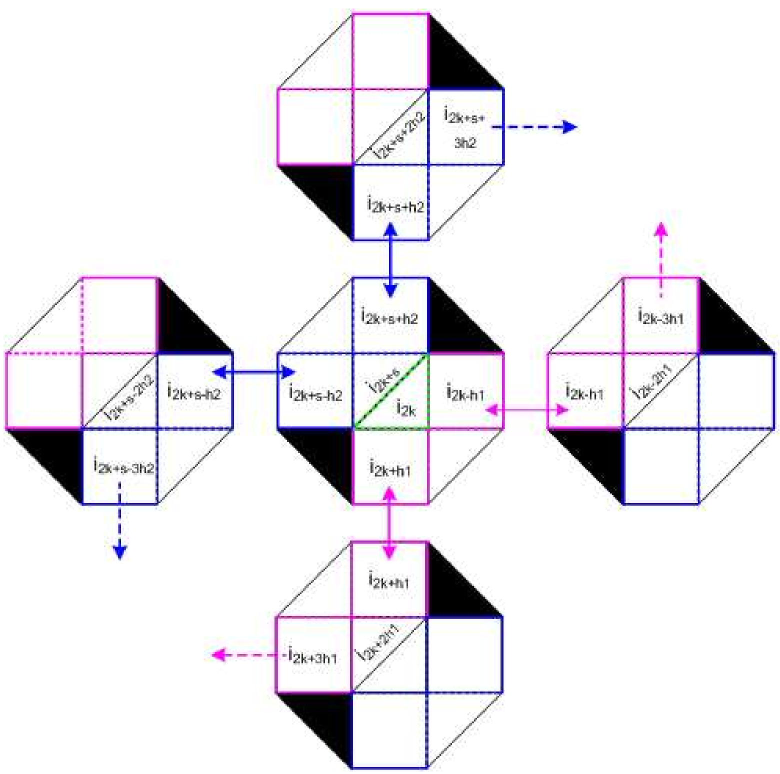
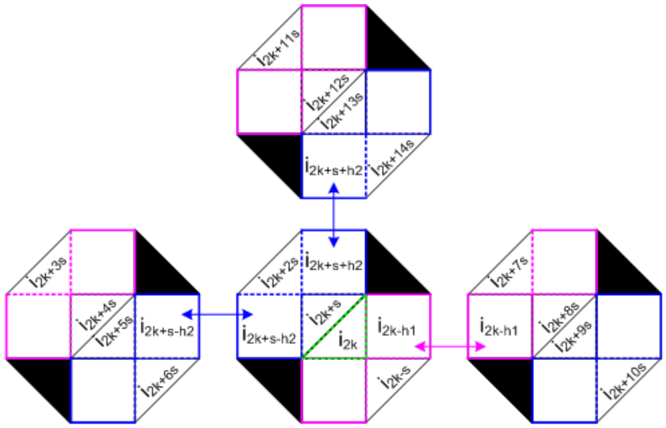


Fig. 2: Triangular prism for chords-ring edges representation



Hamiltonian cycle of ring edges CHRm4(16,1,4,6)

Number	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	0
Ring edges	i_{2k}	i_{2k+s}	i_{2k+2s}	i_{2k+3s}	i_{2k+4s}	i_{2k+5s}	i_{2k+6s}	i_{2k+7s}	i_{2k+8s}	i_{2k+9s}	i_{2k+10s}	i_{2k+11s}	i_{2k+12s}	i_{2k+13s}	i_{2k+14s}	i_{2k-s}	i_{2k}
	i_{2k}	i_{2k-s}	i_{2k+14s}	i_{2k+13s}	i_{2k+12s}	i_{2k+11s}	i_{2k+10s}	i_{2k+9s}	i_{2k+8s}	i_{2k+7s}	i_{2k+6s}	i_{2k+5s}	i_{2k+4s}	i_{2k+3s}	i_{2k+2s}	i_{2k+s}	0

Note: all ring edges must be in *mod N*.
example: i_{2k+s} *mod N*, i_{2k-s} *mod N*

Note: all cell in a same column has same value
example: *1* = $i_{2k+s} = i_{2k-s}$

Fig. 3: Triangular prism for ring edges representation

the chord links for even nodes are $i_{2k} \to i_{2k+h_1}$ and $i_{2k} \to i_{2k-h_1}$ but the chord links for odd number are $i_{2k+s} \to i_{2k+s+h_2}$ and $i_{2k+s} \to i_{2k+s-h_2}$. The edges links for even nodes are $i_{2k} \to i_{2k+s}$ and $i_{2k} \to i_{2k-s}$ and for odd nodes are $i_{2k+s} \to i_{2k+2s}$ and $i_{2k+s} \to i_{2k}$. Let *H* be a Hamiltonian cycle of $\mathrm{CHRm4}(N,s,h_1,h_2)$. Suppose to the contrary that there exist $\overrightarrow{h_1}$ chords and $\overleftarrow{h_1}$ chords or $-\overrightarrow{h_1}$ chords and $-\overleftarrow{h_1}$ chords for even nodes, $\overrightarrow{h_2}$ chords and $\overleftarrow{h_2}$ chords or $-\overrightarrow{h_2}$ chords and $-\overleftarrow{h_2}$ chords for odd nodes. In that case H can be represented as a sequence of $\{s,\overrightarrow{h_1},\overleftarrow{h_1}\}$ or $\{s,-\overrightarrow{h_1},-\overleftarrow{h_1}\}$ or $\{s,\overrightarrow{h_2},\overleftarrow{h_2}\}$ or $\{s,-\overrightarrow{h_2},-\overleftarrow{h_2}\}$ or $\{\overrightarrow{h_1},\overleftarrow{h_1},s\}$ or $\{-\overrightarrow{h_1},-\overleftarrow{h_1},s\}$ or $\{\overrightarrow{h_2},\overleftarrow{h_2},s\}$ or $\{-\overrightarrow{h_2},-\overleftarrow{h_2},s\}$. It's impossible for H to contain $\{\overrightarrow{h_1},\overleftarrow{h_1}\}$, $\{-\overrightarrow{h_1},-\overleftarrow{h_1}\}$, $\{\overrightarrow{h_2},\overleftarrow{h_2}\}$ or $\{-\overrightarrow{h_2},-\overleftarrow{h_2}\}$ as a subsequence. There are so many possibilities for combining s, h_1,h_2,-h_1,-h_2 except $\{\ldots,\overrightarrow{h_1},\overleftarrow{h_1},\ldots\}$, $\{\ldots,-\overrightarrow{h_1},-\overleftarrow{h_1},\ldots\}$,$\{\ldots,\overrightarrow{h_2},\overleftarrow{h_2},\ldots\}$or $\{\ldots,-\overrightarrow{h_2},-\overleftarrow{h_2},\ldots\}$ Fig. 2 shows H can consists of $i_{2k} \to i_{2k-h_1} \to i_{2k-2h_1} \to i_{2k-3h_1} \to \ldots$ or $i_{2k} \to i_{2k+h_1} \to i_{2k+2h_1} \to i_{2k+3h_1} \to \ldots$ for even nodes, while $i_{2k+s} \to i_{2k+s+h_2} \to i_{2k+s+2h_2} \to i_{2k+s+3h_2} \to \ldots$ or $i_{2k+s} \to i_{2k+s-h_2} \to i_{2k+s-2h_2} \to i_{2k+s-3h_2} \to \ldots$ for odd nodes.

The following Theorem 2 shows a CHRm4(N,s, h_1,h_2) contains Hamiltonian cycle if and only if its coprime.

Theorem 2:
CHRm4(N,s,h_1,h_2) contains Hamiltonian cycle if and only if gcd(N,s,h_1,h_2) = 1.

Proof: Let H be a Hamiltonian cycle of CHRm4(N,s,h_1,h_2). CHRm4(N,s, h_1,h_2) = 1 is H if gcd(N,s, h_1) = 1 and gcd(N,s, h_2) = 1 and it is true even for conversion cases.

Corollary1: A CHRm4(N,s,h_1,h_2) must be consists of s = 1. All CHRm4(N,s,h_1,h_2) is a circulant graphs that consists of ring edges 1.

Lemma 2: CHRm4(N,s,h_1,h_2) contains a Hamiltonian cycle at least one of *s*-edges with length 1.

Proof: Triangular prism for ring edges representation was shown in Fig. 3. Let H be a Hamiltonian cycle of CHRm4(N,s,h_1,h_2). Let $0 \le k < \frac{N}{2}$, even node ,i_{2k} will go up or go down along diagonal and continue with diagonal of the other side. H can consists of $i_{2k} \to i_{2k+s} \to i_{2k+2s}$ continued by the other side of $i_{2k+3s} \to i_{2k+4s} \to i_{2k+5s} \to i_{2k+6s}$ continued by the other side of $i_{2k+7s} \to i_{2k+8s} \to i_{2k+9s} \to i_{2k+10s}$ continued by the other side of $i_{2k+11s} \rightarrow i_{2k+12s} \rightarrow i_{2k+13s} \rightarrow i_{2k+14s}$ and lastly continued by the other side of $i_{2k-s} \rightarrow i_{2k}$. The paths can be vice versa.

Asymmetric: Most of previous researchers did their research based on symmetric chordal ring. Symmetric has a property that the network looks same from every node. This property makes this research easier because there is no need to investigate every node because this symmetry allows for identical processors at every node with identical routing algorithms. However CHRm4(N,s,h_1,h_2) is not symmetric graphs or asymmetric and we can observe a small asymmetry between even and odd nodes in CHRm4(N,s,h_1,h_2).

CHRm4(N,s,h_1,h_2) has an objective to construct a large node asymmetric graph with small diameter and offering simple routing algorithms. The following is the definition of asymmetric for CHRm4(N,s,h_1,h_2).

Definition 2: Two nodes that is between even source node, (i_{2k}) and destination node $(i_{2k\pm1})$ or $(i_{2k\pm h_1})$, in a CHRm4 are not similar with odd source node, (i_{2k+1}) and its destination node, $(i_{2k+1\pm1})$ or $(i_{2k+1\pm h_2})$ if for some automorphism α for i_{2k} case of CHRm4, $\alpha((i_{2k})) = i_{2k\pm1}, \alpha((i_{2k})) = (i_{2k\pm h_1})$ and for $(i_{2k\pm1})$ case of CHRm4, $\alpha((i_{2k+1})) = i_{2k+1\pm1}, \alpha((i_{2k+1})) = (i_{2k+1\pm h_2})$ with (i_{2k}), (i_{2k+1}), $(i_{2k\pm h_1})$, (i_{2k+1}), $(i_{2k+1\pm1})$, $(i_{2k+1\pm h_2}) \in N$. Two edges between even case $(i_{2k}, i_{2k\pm h_1})$ and odd case $(i_{2k+1}, i_{2k+1\pm h_2})$ in a CHRm4 are not similar, if for some automorphism α for $\alpha((i_{2k}, i_{2k\pm h_1})) \neq (i_{2k\pm1}, i_{2k+1\pm h_2})$ with $(i_{2k}, i_{2k\pm h_1})), (i_{2k\pm1}, i_{2k+1\pm h_2}) \in h$.

- CHRm4 is called not node-symmetric if every pair of nodes are not similar
- CHRm4 is called not edge-symmetric if every pair of edges are not similar
- CHRm4 is called asymmetric if it is not node-symmetric or not edge-symmetric or both

Every pair of nodes in CHRm4(N,s,h_1,h_2) is not similar between even and odd nodes. Theorem 3 was proven.

Theorem 3: CHRm4 is not node symmetric.

Proof: The connection between two nodes for even case, $(i_{2k}) \rightarrow (i_{2k\pm1}) \rightarrow (i_{2k}) \rightarrow (i_{2k\pm h_1})$ are not similar with the connection between two nodes for odd case, $(i_{2k\pm1}) \rightarrow (i_{2k+1\pm1}) \rightarrow (i_{2k\pm1}) \rightarrow (i_{2k+1\pm h_2})$ in a CHRm4 with $(i_{2k}), (i_{2k\pm1}), (i_{2k\pm h_1})$, $(i_{2k+1}), (i_{2k+1\pm1}), (i_{2k+1\pm h_2}) \in N$. . Hence the proof.

Not every pair of edges in CHRm4 is similar. Theorem 4 was proved.

Every pair of edges in CHRm4(N,s,h_1,h_2) is not similar. Theorem 4 was proven.

Theorem 4: CHRm4 is not edge symmetric.

Proof: Let be two edges $(i_{2k}, i_{2k\pm1})$ and $(i_{2k\pm1}, i_{2k+1\pm1})$ are similar, if for some automorphism α of CHRm4, $\alpha((i_{2k}, i_{2k\pm1})) = (i_{2k\pm1}, i_{2k+1\pm1})$ except $(i_{2k}, i_{2k\pm h_1})$ and $(i_{2k+1}, i_{2k+1\pm h_2})$ are not similar if for some automorphism α of CHRm4, $\alpha((i_{2k}, i_{2k\pm h_1})) \neq (i_{2k\pm1}, i_{2k+1\pm h_2})$ with $(i_{2k}, i_{2k\pm1})$, $(i_{2k}, i_{2k\pm h_1})$, $(i_{2k\pm1}, i_{2k+1\pm1})$, $(i_{2k\pm1}, i_{2k+1\pm h_2}) \in h$.

Hence every pair of edges for even case and odd case is not similar.

CONCLUSION

In this study, we have shown that all the properties are important in CHRm4. Connectivity and asymmetric are the most dominant properties of CHRm4. Connectivity has fulfilled all the distance functions to make sure all nodes were connected to each other. CHRm4 is not node symmetric and not edge symmetric implies CHRm4 as asymmetric structure. Even though CHRm4 is asymmetric but it still gives the best performance compared to CR4. We also prove that CHRm4 must have at least one Hamiltonian cycle that is constructed by ring edges. It is impossible for a sequence of clockwise and anticlockwise of the same chord length to have a Hamiltonian cycle. Further research should explore and develop an optimum routing for CHRm4 and expand it into a broadcasting scheme.

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